Probabilistic Forwarding of Coded Packets for Broadcasting over Networks

A Thesis

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by

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Abstract

Motivated by applications in sensor networks and the Internet of Things (IoT), in this dissertation, we consider the problem of energy-efficient broadcasting from a source node in a large dense network. Flooding, as a broadcast mechanism, involves each node forwarding every packet it receives, to all its neighbours. This results in excessive transmissions and thus a high energy expenditure overall. Probabilistic forwarding involves each node forwarding a received packet to all its neighbours with a certain probability $p < 1$. While this mechanism reduces the number of transmissions, reception of a packet by a network node is not guaranteed.

In the first part of this thesis, we propose a new broadcast algorithm which introduces redundancy, in the form of coded packets, into the probabilistic forwarding protocol to improve the chances of a network node receiving a packet. Specifically, we assume that the source node has k_s data packets to broadcast, which are encoded into $n \geq k_s$ coded packets, such that reception of any k of these coded packets by a network node, suffices to recover the original k_s data packets. Our interest is in determining the minimum forwarding probability, p, for which the expected fraction of nodes receiving at least k out of the n coded packets is close to 1. This we deem a "near-broadcast". The minimum forwarding probability p yields the minimum value for the expected total number of transmissions across all the network nodes needed for a near-broadcast. The expected total number of transmissions is taken to be a measure of the energy expenditure in the network.

In the second part of the thesis, the proposed algorithm is analyzed on deterministic graphs. More specifically, we analyze probabilistic forwarding with coded packets on two network topologies: binary trees and square grids. For trees, our analysis shows that for fixed k, the expected total number of transmissions increases with n. On the other hand, on grids, simulations show that a judicious choice of n significantly reduces the expected total number of transmissions needed for a near-broadcast. It is somewhat counter-intuitive that introducing additional packets in a network can reduce the number of transmissions, but we are able to explain this phenomenon on grids using ideas from percolation theory and ergodic theory. This indicates a benefit in introducing redundancy in the form of coding into the probabilistic forwarding mechanism on grids, but not on trees. The benefit is in terms of a reduction in the overall expenditure of energy in the network to achieve a near-broadcast.

Finally, in the last part of the thesis, we provide an analysis of the performance of the proposed algorithm on random geometric graphs (RGGs). RGGs are used widely to model ad-hoc network deployments. The randomness in the underlying network topology presents additional challenges in the analysis. Our treatment of the problem indicates a trend similar to that on grids: for dense RGGs, with a carefully chosen value of n, it is possible to reduce the expected total number of transmissions while ensuring a near-broadcast, in comparison to probabilistic forwarding with no coding. Our analysis for RGGs involves ideas from Poisson point processes, percolation theory and ergodic theory.

The conclusion we draw from our analysis for trees, grids and RGGs, additionally supported by simulations on several other network topologies, is that on well-connected graphs, there is a benefit to introducing coded packets with probabilistic forwarding.

Publications based on this Thesis

1. B. R. Vinay Kumar and N. Kashyap, “Probabilistic forwarding of coded packets on networks”, *IEEE/ACM Transactions on Networking*, vol. 29, no. 1, pp. 234-247, 2021.

2. B. R. Vinay Kumar, N. Kashyap and D. Yogeshwaran, “An analysis of probabilistic forwarding of coded packets on random geometric graphs”, accepted at *WiOpt 2021*, Philadelphia, October 18-21, 2021. [Extended version]. Available: https://arxiv.org/abs/2105.08779

3. B. R. Vinay Kumar and N. Kashyap, “Probabilistic forwarding of coded packets on networks”, in *Proc. 2019 International Symposium on Information Theory (ISIT 2019)*, Paris, Jul 7-12, 2019.

4. B. R. Vinay Kumar, R. Antony and N. Kashyap, “The Effect of Introducing Redundancy in a Probabilistic Forwarding Protocol”, in *Proc. 2018 National Conference on Communications (NCC 2018)*, IIT- Hyderabad, Feb 25-28, 2018.

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Chapter 1

Introduction

Network sizes have grown in recent years owing to cheap electronics available in the market. Ad-hoc networks which have no centralized infrastructure have become ubiquitous. The main example is that of wireless sensor networks (WSNs) which have been deployed for applications ranging from agriculture, industry and military reconnaissance.

The advent of the Internet of Things (IoT) in recent times has introduced a new dimension to the study of ad-hoc networks. Our everyday life is replete with numerous IoT devices and sensors whose primary function is something other than communication. For example, sensors and actuators in a car are geared towards control and movement of the car. These are integrated into a network to provide additional functionality, such as self-driving ability. Such scenarios limit the amount of energy available to the IoT node for communication.

More generally, nodes in an ad-hoc network can be thought of as functioning with the following constraints.

- **Energy constraint:** Ad-hoc networks are equipped with nodes which have a limited battery capacity. Similarly, nodes of a WSN deployed in a field are typically energy harvesting nodes. These have limited energy to expend for transmission of data packets.

- **Computational constraint:** As mentioned before, nodes in an ad-hoc network have a functionality that is different from communication. Owing to this and the energy constraint, nodes are usually not equipped with computational resources which can aid in communication.

- **Knowledge constraint:** The nodes in the network have no knowledge of the overall network topology. Additionally, they do not have any information about their relative position in the network.

Furthermore, in the case of IoT networks, the sheer number of devices form a large and dense network with heterogeneous nodes. In such scenarios, erstwhile algorithms for information dissemination and communication need to be rethought to account for these new challenges.

Broadcast mechanisms are vital to disburse key network-related information in such ad-hoc networks. For example, updating the sensing parameters in WSNs and over-the-air programming of IoT nodes are typically done through a broadcast mechanism. These broadcasts are usually initiated from a single node in the network which is easily accessible (a mobile phone, say).

Our primary goal is to disseminate information from the source node in a distributed network with minimal energy expenditure. In this thesis we propose and analyze a broadcast algorithm which conforms to the constraints imposed on the individual nodes. Before we describe the algorithm, we first present a case study of various practical deployments of WSNs which elucidate the constraints on the individual nodes.

1.1 Case study: Deployments of wireless sensor networks

In 1965, Gordon E. Moore predicted a doubling of the number of components per integrated circuit annually, in his paper [2]. One of the leading contributors towards this increase in the past four decades has been the rapid progress in the communication industry, as evidenced by numerous articles (see e.g., [3–5]). In [6], the authors comment

"*At the end of 2008, more than 4 billion mobile phones were estimated to exist world-wide, representing more than* 60% *of penetration. Another emerging market of wireless sensor networks will tend to grow significantly in the next years which can already reach approximately 120 million of remote units by 2010*"

The evolution of micro-electro-mechanical systems (MEMS) technology brought about easy and fast production of cheap sensor nodes. Coupled with communication technologies such as Bluetooth, ZigBee, WiFi, RFID, Visible Light Communication etc. during the turn of the century, sparked a revolution in the state of the art in WSNs. In recent times, they have been deployed ubiquitously for applications ranging from military reconnaissance, environment monitoring, health tracking etc. (see e.g., [7]). We provide examples of few such deployments which bring out the challenges associated with information dissemination in these wireless sensor networks.

In [8] ,the authors develop an online microclimate monitoring and control system for greenhouses. They field-test the system in a greenhouse in Punjab, India, evaluating its measurement capabilities and network performance in real time. The authors in [9] use image processing sensor nodes to monitor a vineyard for different types of deficiency, pests or diseases. Once pests are detected, their location has to be communicated rapidly for quick isolation. In [1], the authors monitor trees in a 13×40 date-palm orchard in Israel by deploying sensor nodes. The nodes form a network with a central coordinator which collects the data. An illustration from their paper is shown in Fig. 1.1.

Similar technological solutions using sensor networks for agriculture have been developed for irrigation in [10–12]. [13] provides a review of specific issues and challenges associated with deploying WSNs for improved farming. Most of these networks are spatially distributed and the individual nodes are typically wireless nodes deployed to form a network. Moreover, since farming is done in a pattern with prescribed gaps between rows of crops, one can expect that the associated sensor network deployed also has a regular lattice kind of topology.

Environment monitoring has been another area where WSNs have been widely used. The authors in [14] provide an extensive review of some of the practical implementations.

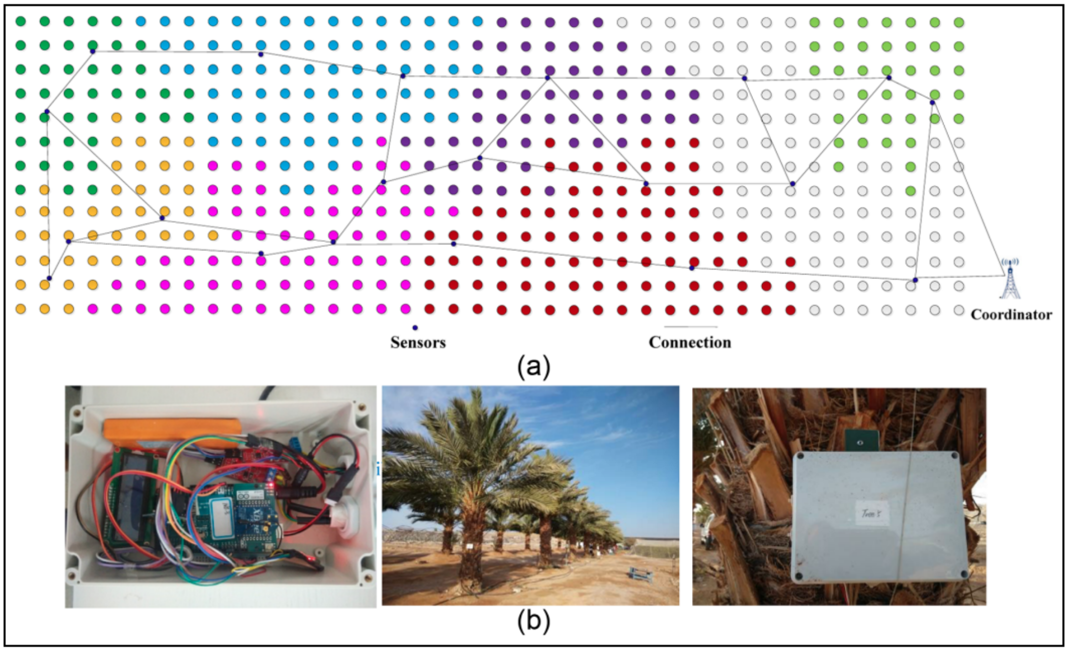


Figure 1.1: Sensors deployed in a date-palm orchard of 520 trees. Picture taken from [1].

In [15, 16], the authors describe wireless sensor networks for flood warning. They collect water level measurement data on remote locations that are covered by surface waters, either by flooding or seasonal environmental impacts. Places affected by floods frequently, may be inaccessible and hence the sensor nodes deployed in such regions need to be robust and durable. In [17], energy-harvesting nodes are employed to make the nodes self-sustaining. The authors in [18] provide a review of the state-of-the-art in energy-harvesting WSNs for environmental monitoring applications. Air quality monitoring in industrial and urban areas using Zigbee WSNs has been proposed and implemented in [19]. IoT solutions with low-cost systems and a large number of sensors have also been proposed in [20]. Zigbee and IoT standards restrict the devices to be low-power, low data rate, and within close proximity of each other. Nodes are typically arranged in a mesh network when using these technologies.

In recent times, a diverse set of applications in the indoor environment which employ WSNs have cropped up. The IoT framework has further bolstered such applications due to its ease of adaptability and implementation. As part of Industry 4.0, supply chain management and smart logistics have adopted information and communication technologies to improve on the efficiency while incurring minimum cost overhead. [21] provides a review of different works that have gone into streamlining the logistics industry including package transportation, warehousing, loading/unloading, distribution etc. Design of intelligent warehouses and their management has been automated (see e.g., [22–24]) and there are companies similar to Digiteum [25] which offer solutions to make the warehouses smarter. A combination of different technologies are typically used in these applications. Additionally, to optimize the space in the indoor environments, sensors are deployed in close proximity of one another resulting in dense networks.

From these use cases, it is clear that WSNs form large and dense networks with the individual nodes being energy constrained with minimal computational resources. They are decentralized and are sometimes arranged in a regular lattice-like underlying topology. Information dissemination algorithms when deployed on such networks, need to account for these constraints on individual nodes.

1.2 Motivation

A typical motivating example that one can have in mind for the purpose of this thesis is a network of temperature or humidity sensors that are deployed in a field as in Fig. 1.2. These could be used to monitor the soil and weather conditions to take appropriate control actions when necessary. Wireless sensor networks (WSNs) in such outdoor environments typically consist of energy-harvesting nodes. Since these are spatial networks over a large geographical area, they have a central station which is easily accessible; possibly even a mobile phone. Call this the source s. Network-critical information, such as the sensing frequency, or the firmware are updated from time to time in these networks. Dissemination of such information to all the other nodes in the network happens through a broadcast mechanism.

A natural broadcast algorithm is flooding, wherein a node forwards every newly received packet to all its one-hop neighbours. However, a node might receive the same packet from multiple neighbours resulting in wasteful transmissions. Moreover, flooding

Figure 1.2: Sensor nodes deployed in a field. Node s is the source.

is also known to result in the 'broadcast-storm' problem [26]. In short, although the flooding mechanism is simple and easy to implement, there is an excessive number of transmissions in the network, resulting in a high energy expenditure.

For the applications that we are interested in, such a broadcast algorithm is not feasible. To adhere to the constraints mentioned in the previous section, any broadcast algorithm that is proposed needs to have the following characteristics:

- **Completely distributed:** The nodes in the network do not have any knowledge of the network topology. Owing to this, they need to make decisions about forwarding packets independently of other nodes. Moreover, trying to learn the network structure involves additional transmissions which is undersirable. Thus, the broadcast algorithm needs to be completely distributed and decentralized.

- **Minimal energy consumption:** A large part of the energy available for communication in a node is utilized while transmitting information. Minimizing the energy consumption is equivalent to minimizing the number of transmissions in the network. Thus, the broadcast from the source should reach all the nodes in the network with minimal number of transmissions.

- **Limited computation:** The broadcast algorithm must impose minimal computational burden on the individual nodes.
- **Finite execution time:** On a network with a finite (but possibly large) number of nodes, the broadcast algorithm should terminate in finite time.

Probabilistic forwarding (or probabilistic retransmission) as a broadcast mechanism, has been proposed in the literature (see [27]) as an alternative to flooding. Here, each node, on receiving a packet for the first time, either forwards it to all its one-hop neighbours with probability p or takes no action with probability $1 - p$. While it is evident that probabilistic forwarding uses lesser number of transmissions compared to flooding, it has the drawback that a particular node in the network may not receive a packet, and hence, is unable to obtain the information from the source.

In this thesis, we propose to introduce additional coded packets along with probabilistic forwarding in order to alleviate this problem. These are defined in the next section. Error / erasure correction capability of the code is used to guard against unavailability of a packet at a node due to probabilistic forwarding. While it might seem that introducing such coded packets increases the number of transmissions in the network, we will show the following very counter-intuitive result. On most well-connected graphs, such as grids and random geometric graphs (in the super-critical region), probabilistic forwarding with coding leads to a decrease in the number of transmissions as compared to the case with no coding for carefully chosen values of the number of coded packets and the forwarding probability. However, this is not true in the case of broadcasting on trees, i.e., coding is not beneficial in terms of the number of transmissions required for a broadcast on trees.

Our main goal in this thesis is to analyze the proposed mechanism on dense random geometric graphs since these are used to model ad-hoc networks. In the process, we find that understanding the mechanism on deterministic graph topologies such as trees and grids is essential. Random geometric graphs are locally tree-like when the intensity of points is low. When the intensity is high the probabilistic forwarding mechanism exhibits similar trends as of a grid. We will see that understanding our proposed mechanism on these graphs provides us techniques and intuitions necessary to explain the behaviour of our mechanism on random geometric graphs.

In Section 1.3, we include a brief review of the literature which tackle similar problems or have similar broad motivation as us.

1.3 Related Work

Algorithms for broadcast over ad-hoc networks have garnered considerable attention in the past. We refer the reader to [28], [29] and [30] and the references therein for a review of the broad categories of algorithms employed for broadcasting. We further supplement this list with references relevant to our work here. The broadcast algorithm proposed and analyzed in this dissertation is an amalgamation of probabilistic forwarding along with encoding of packets at the source. In the following, we highlight relevant literature from these two areas.

1.3.1 Coding based approaches

Network coding

Network coding has been used for efficient data dissemination in wireless networks in [29–34]. In [31], the authors propose random linear network coding (RLNC) for the multicast problem and give bounds on the probability that all the receivers are successful in obtaining the packets. The authors in [33] compare the number of transmissions in the RLNC based approach with that of store-and-forward approaches (which includes probabilistic forwarding) on a circular network topology. Network coding schemes are shown to be energy-efficient. Similar deductions are made via simulations in [34] for employing network coding in a medical sensor network. In [32], the authors provide transmission strategies for universal recovery and arrive at necessary and sufficient conditions on the number of transmissions required using network coding. However they assume complete knowledge of the network topology at every node.

Our work is closest in spirit to that in [29], where the authors have a similar motivation as ours, namely, to propose a low-complexity distributed broadcast algorithm, with nodes having no prior knowledge of the network topology. Moreover, similar to our considerations, their figure of merit is energy efficiency which is quantified using the number of transmissions required for the broadcast. They employ network coding and propose a decentralized algorithm that improves upon the number of transmissions in flooding by a constant factor. While our work also addresses similar questions, the results in the two works are not directly comparable. In the setting of [29], all the nodes in the network have messages to broadcast, making the network coding approach attractive. On the other hand, in our setting, only a single source node has messages that need to be broadcast.

Other coding schemes

Unlike network coding schemes, in our work, packets are encoded only at the source before transmission. The class of codes that we propose, includes among others, fountain codes which have been used widely in broadcast mechanisms for ad-hoc networks. This is primarily because they form a convenient alternative to the ARQ (Automatic repeat request) protocol. In the ARQ scheme, an acknowledgement (ACK) needs to be sent every time a packet is received. By employing fountain codes, a node is required to send an ACK less frequently, thus saving on energy.

The authors in [35] employ fountain codes for broadcasting in vehicular networks. However, unlike our setting, all the nodes are in a star topology and receive transmissions from the source through erasure channels. In [36, 37], the authors use Luby transform (LT) codes, a special case of fountain codes, which reduces the complexity of encoding and decoding at the network nodes. The LT encoding is done by randomly selecting d packets from n packets and doing an XOR of these packets to form a single encoded packet. The authors in [36] propose to employ LT codes in conjunction with transmission over a source-independent backbone network. They show via simulations that this approach not only reduces the number of transmissions required for flooding, but also reduces the packet delay. The variable d is an integer which is chosen according to a distribution. In [37], the authors propose a new distribution on d which further brings down the delay and the number of transmissions. However, both these approaches require the knowledge of a dominating set which is a subset of nodes of the network such that every node in the network is either in this set or adjacent to a node of this set. Finding a dominating set is computationally expensive. In [38], the authors construct novel codes called rateless online MDS (ROME) codes for wireless broadcasting. They are shown to have lesser coding redundancy and number of transmissions as compared to LT codes. However, they exploit feedback information from the receivers.

1.3.2 Probabilistic forwarding based approaches

Probabilistic forwarding as a broadcast mechanism (see e.g., [27]) has been proposed in the literature as an alternative to flooding. Here, each node, on receiving a packet for the first time, either forwards it to all its one-hop neighbours with probability p or takes no action with probability $1-p$. An excellent summary of the recent literature on probabilistic broadcast mechanisms is provided in [39, Chapter 3].

GOSSIP algorithm and variants

Probabilistic forwarding, as described in Section 1.2, has also been referred to as the GOSSIP1(p) algorithm in [40]. The authors claim a 35% reduction in the transmission overhead as compared to flooding. Further, several variants of the probabilistic GOSSIP1(p) protocol are described and heuristics and simulation results are provided for improving flooding and routing mechanisms in networks. The variants include:

- GOSSIP1(p, k) - Transmit with probability 1 upto k hops from the source, followed by probabilistic forwarding with probability p by nodes further away.

- GOSSIP2(p_1, k, p_2, n) - The first two parameters are as in the GOSSIP1(p, k) protocol. The new features are p_2 and n; the idea is that the neighbours of a node with fewer than n neighbours gossip with probability $p_2 > p_1$. That is, if a node has fewer than n neighbours, it instructs its immediate neighbours to broadcast with probability p_2 rather than p_1.

- GOSSIP3(p, k, m) - Here again, the first two parameters correspond to the GOSSIP1 protocol. A node that originally did not broadcast a received message, but then did not get the message from at least m other nodes within some timeout period, broadcasts the message immediately after the timeout period.

There have been numerous other works, for example see [41–45], which propose improvements on the GOSSIP protocol. In [42], the authors target a similar problem as ours: achieve a high degree of network coverage with limited number of transmissions. They even employ very similar analytical techniques based on continuum percolation to characterize two gossip algorithms: global gossip and distributed gossip. However, they assume some knowledge of the average degree of the random planar network at every node of the network. The authors in [46] propose a novel approach to combine tree-based and gossip protocols in order to achieve both low message complexity and high reliability. Hypergossiping has been proposed in [43] to overcome problems of connectivity in mobile ad-hoc networks. In [45], the authors propose the smart gossip protocol which aims to adaptively set the forwarding probability at each node by quantifying the "importance" of each node for achieving dissemination. However, all of these works evaluate the proposed algorithm using extensive simulations and lack sound analytical characterization.

Choice of forwarding probability

A significant portion of the literature on probabilistic forwarding dwells upon setting the forwarding probability based on different notions. We have a similar motivation in this thesis as well. In the following we highlight a few such notions.

- **Neighbour based approaches**: In these schemes, the forwarding probability is decided based on the number of neighbours or the density of nodes in a region. The main rationale behind this approach is that, higher the density or the number of one-hop neighbours, lower the forwarding probability. In [47], the authors use a forwarding probability proportional to the inverse of the number of neighbours a node has. The authors in [48, 49] use a similar idea but with additional deterministic corrective measures to improve on the probabilistic scheme. A dynamic adaptive scheme to set the forwarding probability is discussed in [50] for mobile ad-hoc networks and is compared and shown to be better than fixed probability schemes via simulations. Slightly more sophisticated algorithms are explored in [51] and [52] which use a neighbour coverage-based probabilistic rebroadcast protocol by introducing a delay for rebroadcast. However this involves additional resources at each node in the network.

- **Area/distance based approaches**: In area-based schemes, the forwarding probability is set based on an estimate of the additional area that will be covered by a node if it transmits. This additional area is estimated based on either the number of copies a node receives or the distance from the node whose transmission it receives. In [53], three algorithms are proposed based on these ideas: Area Coverage-based Probabilistic Forwarding (ACPF), exploits the overlapping of transmission areas between neighbouring nodes to set a higher value of the forwarding probability when the coverage areas is large. The second scheme, referred to as Copies Coverage-based Probabilistic Forwarding (CCPF), uses the number of duplicate request messages overheard during a random time interval to determine the forwarding probability, p. As the number of the overheard duplicate request messages increases, the forwarding probability of the node decreases. The third scheme, referred to as Area and Copies Coverage-based Probabilistic Forwarding (ACCPF), takes advantage of both the transmission area coverage and the number of the overheard duplicates of the same request to determine the value of p. In [54], a technique similar to CCPF is used to infer the density of nodes in a neighbourhood and thus choose a smaller forwarding probability if the node is in a dense region.

  In [55], each node chooses a forwarding probability which is proportional to the distance from which it received the packet. The relative distances between nodes are assumed to be known beforehand at each node. Variations of this algorithm where the probability used is proportional to some k-th power of the relative distance between nodes is discussed in [56].

- **Interference based schemes**: In this scheme, nodes in the network choose a forwarding probability based on the signal strength with which they receive packets. For example, in [57], the nodes obtain a signal-to-interference-plus-noise ratio (SINR) value as measured at the physical layer. If the SINR is low, it means that surrounding nodes may not have received the packet and hence the node chooses a higher probability of retransmission. In [58], the gossip probability is chosen based on the received signal strength (RSS). RSS is an indication of the channel quality, and hence nodes transmit with higher probability when the channel is good.

There are numerous other approaches which combine different methods to set the forwarding probability. The interested reader is referred to the survey paper [59] and Chapter 3 of [39]. However, there are two main differences between these works and ours. Firstly, such schemes require some knowledge about the network topology either in terms of the number of neighbours or distance from a nearest node etc., which we do not assume in our work. Secondly, and more importantly, most of these are simulation based studies with no analytical backing. Our aim in this thesis is to provide a robust analytical framework to the algorithm we propose which can perhaps be extended to analyze some of these algorithms as well.

Other variations of probabilistic forwarding

The authors in [60] map randomized broadcast mechanisms to percolation on networks, which is the approach we take in this thesis as well. They, however, use directional antennas to reduce the transmission overhead and map it to a bond percolation problem. In [61], the authors propose Robust Probabilistic Flooding mechanism which takes into account the energy-harvesting nodes and the times they are active. The works in [62] and [63] consider broadcast problems on topologies similar to ours but a different mechanism. In [62], the authors model each edge of a tree as a binary symmetric channel and aim to recover the data present at the root of the tree using information from the nodes at level ℓ. Similar considerations are discussed on an infinite directed acyclic graph with the form of a $2D$ regular grid in [63].

1.3.3 Situating our work

Our main goal is to propose a simple, distributed, light-weight broadcast algorithm for ad-hoc networks. We combine ideas from both the coding based approach as well as the probabilistic forwarding approach. More specifically, we introduce redundancy in the form of coded packets into the probabilistic forwarding mechanism. The randomness brought about by the probabilistic forwarding algorithm can be compensated by the structural properties of the code we employ. To the best of our knowledge, we are the first ones to propose such an algorithm. Moreover, it should be highlighted here that, it is the analysis of the proposed algorithm which is far more valuable to the field, since many models which have been proposed lack a solid theoretical foundation.

1.4 Organization of this thesis

The dissertation is divided into three parts. In the first part, we describe the probabilistic forwarding mechanism with coded packets. We explain our problem setup and establish a formal problem statement (Chapter 2). Simulations of the proposed broadcast mechanism are carried out on different graph structures (Chapter 3) and some initial deductions are drawn from the observed behaviour (Chapter 4).

The second part of the thesis analyzes the proposed broadcast mechanism on deterministic graphs. The focus is on trees (Chapter 5) and grids (Chapter 6). The analysis on trees involves concentration results for random variables with Binomial distribution and the analysis on grids is via percolation theory and ergodic theory.

In the last part of the thesis, the probabilistic forwarding mechanism is analyzed on random graph topologies. Random geometric graphs (RGGs) are a class of random graphs which are used to model practical deployments of ad hoc networks such as those discussed in Section 1.1. The theoretical characterization of the probabilistic forwarding mechanism on RGGs (in Chapter 7) builds upon the ideas from the analysis on grids. Ideas from Poisson point processes, continuum percolation and ergodic theory are used. In Chapter 8, the probabilistic forwarding mechanism is investigated on random regular graphs (RRGs). Preliminary results, using a generating function approach, are obtained to justify the trends on these graphs in a restricted regime.

The thesis concludes with a chapter summarizing the contributions and exploring some directions that can be pursued in the future (Chapter 9). The Appendix collects some auxiliary results needed in Parts II and III of the thesis.

Some general notations

Symbol	**Meaning**
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$[n]$	$\{1, 2, \cdots, n\}$
$\nu(\cdot)$	Lebesgue measure on $\mathbb{R}^2$
k_s	Data packets that source s intends to broadcast
k	Minimum packets to be received for correct decoding
n	Number of coded packets
$p_{k,n,\delta}$	Minimum forwarding probability
$\tau_{k,n,\delta}$	Expected total number of transmissions
$\mathcal{R}_{k,n}$	Nodes that receive at least k out of the n coded packets- successful receivers
$R_{k,n}$	Number of successful receivers
Γ_m	$\left[\frac{-m}{2}, \frac{m}{2}\right]^2$ for $m \in \mathbb{R}$. Square area around the origin in $\mathbb{R}^2$
Λ_m	$\left[-\frac{m-1}{2}, \frac{m-1}{2}\right]^2 \cap \mathbb{Z}^2$ for m odd integer. Discrete grid around the origin.

Part I

Problem Setup

Chapter 2

Problem setting

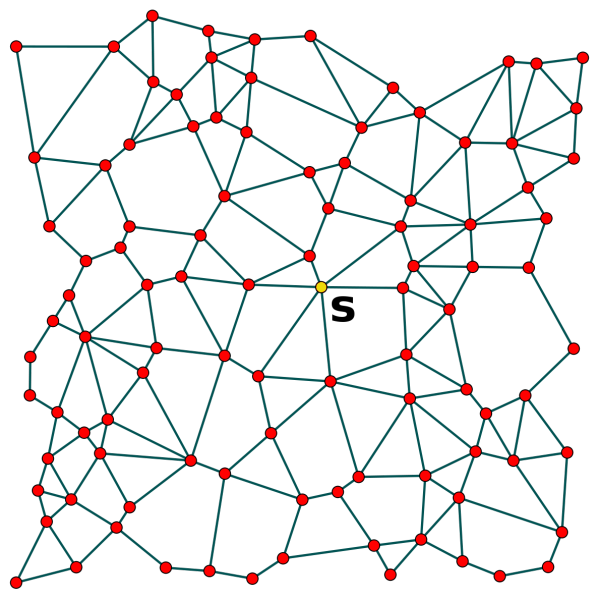


Figure 2.1: Graph G with source s.

Consider a connected graph $G = (V, E)$, where V is the vertex set with N vertices (nodes) and E is the set of edges. Since we operate in a wireless medium, the nodes adjacent to vertex u in the graph are those that can receive a transmission from u. It is assumed that when a node broadcasts a packet, all its one-hop neighbours receive the packet without any errors. In other words, the edges of G are all noiseless communication links. A particular node is distinguished as the source s in the graph, which possesses information that is to be broadcast.

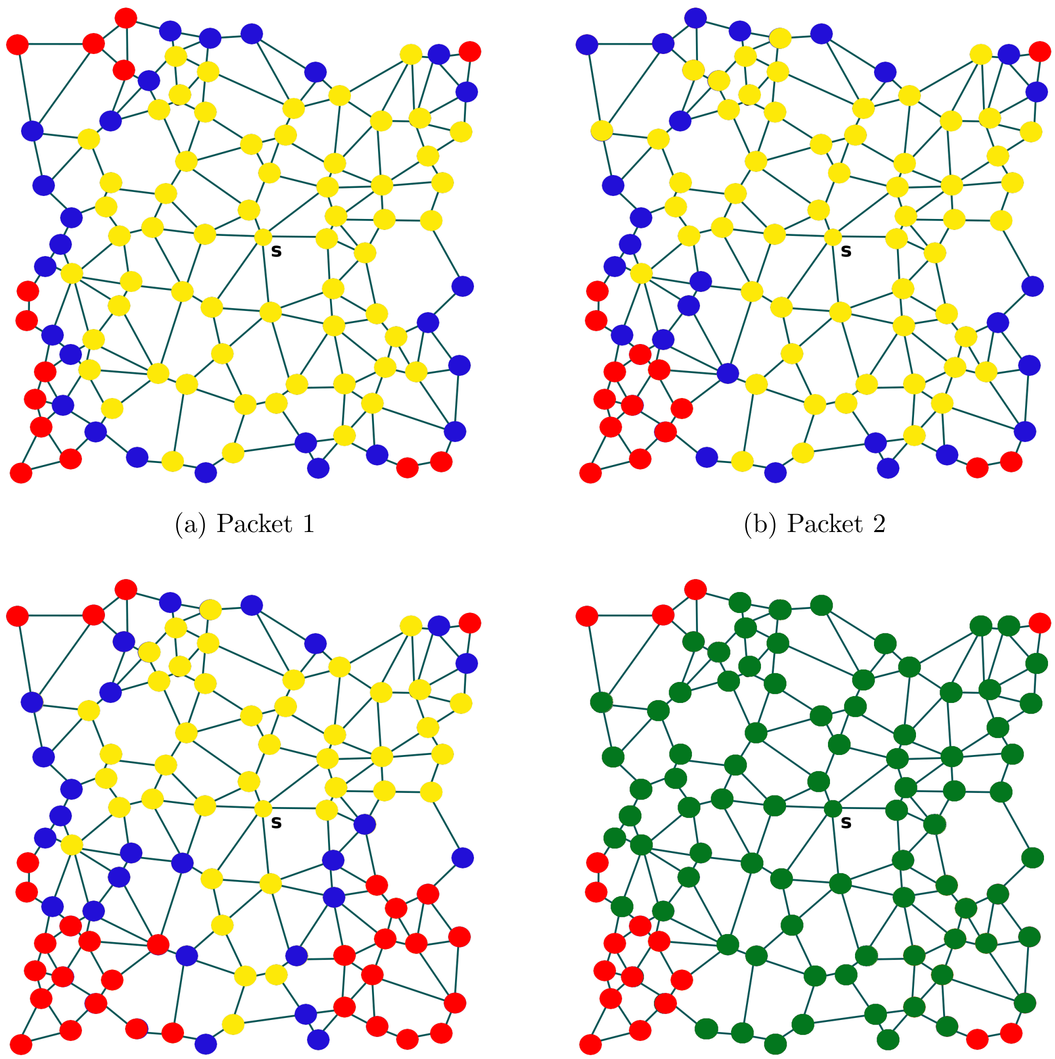
Coding mechanism: The source s has k_s message packets which need to be broadcast in the network. The k_s message packets are first encoded into n coded packets such that, for some $k \geq k_s$, the reception of any k out of the n coded packets by a node suffices to retrieve the original k_s message packets. Examples of codes with this property are Maximum Distance Separable (MDS) codes ($k = k_s$), fountain codes ($k = k_s(1 + \epsilon)$ for some $\epsilon > 0$) etc. which are used in practice. We assume that all the required encoding/decoding operations are carried out over a sufficiently large field, so that codes with the necessary parameters exists.

Transmission scheme: The n coded packets are indexed using integers from 1 to n, and the source transmits each packet to all its one-hop neighbours. All the other nodes in the network use the probabilistic forwarding mechanism: when a packet (say, packet $\#j$) is received by a node for the first time, it either transmits it to all its one-hop neighbours with probability p or does nothing with probability $1 - p$. This decision by a node to forward packet $\#j$ is made independently of the other nodes and other packets. The node ignores all subsequent receptions of packet $\#j$, irrespective of the decision it took at the time of first reception. Packet collisions and interference effects are neglected.

In this thesis, we will refer to the broadcast scheme described here as the *probabilistic forwarding* or the *probabilistic retransmission* mechanism without explicit reference to the coding mechanism. It is to be understood that the scheme operates on the n coded packets.

2.1 Problem formulation

Let $\mathcal{R}_{k,n}$ be the nodes, including the source node, that receive at least k out of the n coded packets. Owing to our coding scheme, it is these nodes which can decode the information contained in the k_s message packets transmitted by the source. We call these *successful receivers* and denote the number of such nodes by $R_{k,n}$. An example illustration with $k = 2$ and $n = 3$ is shown in Fig. 2.2.



(a) Packet 1

(b) Packet 2

(c) Packet 3

(d) Successful receivers ($\mathcal{R}_{2,3}$)

Figure 2.2: Illustration of probabilistic forwarding with 3 coded packets. Here the yellow nodes (●) receive the packet from the source and transmit it. The blue nodes (●) receive the packet but do not transmit it. The red nodes (●) do not receive the packet. The green nodes (●) are those that receive at least 2 out of the 3 coded packets. They are the successful receivers.

Since we are interested in a broadcast scheme, we want the expected fraction of successful receivers to be close to 1. This, we deem a "near-broadcast". More formally, given a $\delta \in (0,1)$, let $p_{k,n,\delta}$ be the minimum forwarding probability p for a near-broadcast, i.e.,

$$p_{k,n,\delta} \ := \ \inf \left\{ p \ \middle| \ \mathbb{E}\left[\frac{R_{k,n}}{N}\right] \geq 1-\delta \right\}. \tag{2.1}$$

For a fixed (deterministic) underlying graph, the expectation above is with respect to the randomness in the probabilistic forwarding mechanism. However, for random graphs, the randomness in the graph should also be considered. The broadcasting problem only makes sense if the underlying graph is connected. On random geometric graphs (RGGs), which are of primary interest to us, we take the connected component containing the source node as the graph, G, over which we employ our broadcast algorithm. In Chapter 8, we discuss in brief, probabilistic forwarding on random regular graphs (RRGs). Here again, we consider the connected component of the source as the graph over which probabilistic forwarding is implemented. In both these cases, the value of N is the total number of nodes within this component of the source, which is also a random quantity. The expectation above is then, not just with respect to the probabilistic forwarding mechanism, but is also over the realizations of this graph, G.

On other random graph models, the expectation in (2.1) needs to be defined on a case-by-case basis. For example, on models with a fixed number of vertices, the expectation can be taken over a probability distribution on all the connected realizations of the graph in addition to the randomness in the probabilistic forwarding mechanism. This could, for instance, be achieved by conditioning on the event that the random graph is connected.

The quantity $p_{k,n,\delta}$ in (2.1), more plainly, is the minimum probability with which each node in the network needs to forward a packet, so that a large (expected) fraction of nodes receive the information from the source. The performance measure of interest, denoted by $\tau_{k,n,\delta}$, is the expected total number of transmissions across all nodes when the forwarding probability is set to $p_{k,n,\delta}$. Here, it should be clarified that whenever a node forwards (broadcasts) a packet to all its one-hop neighbours, it is counted as a single (simulcast) transmission.

Our aim is to determine, for a given k and δ, how $\tau_{k,n,\delta}$ varies with n, and the value of n at which it is minimized (if it is indeed minimized). To this end, it is necessary to first understand the behaviour of $p_{k,n,\delta}$ as a function of n. Our primary interest is to characterize the probabilistic forwarding mechanism on random geometric graphs (RGGs) which are described in Chapter 3. These are used to model ad-hoc networks such as those in Section 1.1.

The quantities $R_{k,n}$, $p_{k,n,\delta}$, $\tau_{k,n,\delta}$ etc. are all, of course, functions of the underlying graph G as well, but for simplicity, we usually suppress this dependence from our notation. We use $R_{k,n}(G)$, $p_{k,n,\delta}(G)$, $\tau_{k,n,\delta}(G)$ etc. whenever the dependence on G needs to be made explicit.

2.2 Problem variants

In this section we discuss variants of the proposed probabilistic forwarding mechanism with coded packets. We discuss their feasibility and additional assumptions required on the individual nodes.

2.2.1 Coding at intermediate nodes

A natural light-weight extension to the algorithm proposed here is when nodes other than the source are also allowed to encode packets. A node upon receiving at least k out of the n packets can decode the information contained in the k_s source packets. This node can now act as a secondary source. It can encode the k_s packets again and transmit them further.

A few challenges that arise in this approach are to decide which nodes act as secondary sources and the forwarding probability that nodes should choose. Nodes which are near to the source receive a large number of packets and can act as secondary sources. However, regarding all of them as secondary sources will be wasteful. A possible rule of thumb to overcome this is to stipulate that, only those nodes which receive exactly k out of the n packets act as secondary sources. Such nodes can be expected to be present at the boundary of the cluster of successful receivers. In other words, nodes which receive exactly k out of n packets can be expected to have neighbours which receive less than k packets (or which are not successful). Additionally, one can reduce the forwarding probability of these secondary sources. Naturally, this is a harder problem to analyze, and we believe that the analysis in this thesis will prove to be a stepping stone in understanding such algorithms.

2.2.2 Varying forwarding probability

Another variation of the proposed mechanism is to have different forwarding probabilities at different nodes. A node could transmit a packet based on its distance from the source. This requires a knowledge of the distance from the source to be made available at the node. A similar idea has been proposed in [64] for vehicular networks, where the authors show, via extensive simulations, that a well-chosen forwarding probability reduces the delay in the network while having a high success probability of broadcasting. However, they assume knowledge of the distance from the source and local neighbourhood information. An easy way to make this information available at the node is to include a counter in the header of each packet which gets updated as the packet traverses the network.

On trees, where there is a unique path from the source to any node in the graph, including such a counter provides an accurate value of the distance from the source at the node. The node can choose an appropriate forwarding probability with this knowledge. We discuss this in more detail in Section 9.1.3. On more well-connected graphs such as lattice structures, a packet from the source may reach a node through a long convoluted path. Then, the value of the distance counter is not accurate which reflects in the node's choice of the forwarding probability. This results in additional transmissions in the system.

Multiple packet transmissions can be exploited to obtain better estimates of the distance of a node from the source. Inferring the minimum distance from the source, in a lattice topology, can also be viewed through the lens of first and last passage percolation (see e.g., [65], [66]). These result in interesting questions but we do not discuss them in this thesis.

Alternately, a node can decide the forwarding probability based on the number of copies it receives of a particular packet. Reception of the same packet from multiple neighbours indicates that the node has numerous neighbours. Its transmission of the same packet may not help in increasing the number of receivers. While this is a feasible approach, it requires the node to keep track of all the nodes from which it received a packet.

2.2.3 Other variants and comments

Several other variants of the probabilistic forwarding protocol can be conceived and implemented. A few are listed below.

- Maintain a list of packets that have been received but not forwarded and choose one among them to transmit. Additionally, a node could use more sophisticated means of deciding which received packets it should forward.

- Learn the local network structure based on packet receptions and tune the forwarding probability accordingly.

- Broadcast using directional antennas in a direction opposite to the one received from.

These algorithms, however, require either greater knowledge of the network topology, or they demand additional resources such as buffers or computation capability at the individual nodes. This does not align with our idea of a completely distributed, energy-efficient broadcast algorithm.

2.3 Communication aspects

In this section, we consider some of the issues involved in the practical implementation of the proposed probabilistic forwarding algorithm with coded packets. With numerous packets traversing the network, packet collisions are bound to happen. These interference effects need to be handled. Moreover, the channel between adjacent nodes could be error prone resulting in a transmission being lost. Such channel outages need to be addressed as well.

When there are multiple packets in the network, interference effects can be avoided by separating the transmissions either in the frequency domain or the time domain.

- In the frequency domain, a possible solution is for nodes to transmit on orthogonal sub-carriers of an Orthogonal Frequency Division Multiplexing (OFDM) signal. Alternately, each packet could be transmitted on a different orthogonal sub-carrier. The latter scheme, however, limits the number of packets that can be transmitted concurrently.

- In the time domain, a scheduling algorithm has to be implemented to avoid concurrent transmissions which might interfere. A schedule must not have a pair of nodes that are within two hops from each other in the same slot.

From a graph theoretic perspective, both these solutions can be viewed as vertex-colouring problems on the underlying graph $G = (V, E)$. The problem of *broadcast scheduling* captures this from a graph colouring setting and has a vast literature (see e.g., [67, 68]). Two vertices $u, v \in V$ can have the same colour if and only if both of the following conditions hold.

- $e = (u, v) \notin E$.

- There does not exist a vertex x such that $e_1 = (u, x) \in E$ and $e_2 = (x, v) \in E$.

This problem has been shown to be NP complete in [69]. Several centralized and distributed algorithms have been proposed in the literature for obtaining a schedule through vertex-colouring (see [68, Table 1]). A heuristic, but centralized solution in [70], called RAND, is known to give very efficient slot schedules. In [71], the authors introduce DRAND, a distributed, robust and scalable implementation of RAND which is also very simple and easy to implement in practical systems. Moreover, it does not require clock-synchronization or global information. Another distributed algorithm called *distributed scheduling using topological ordering* (DSTO) has been proposed recently in [68]. The algorithm is scalable in a fully distributed manner and reduces running time and message overhead. Once a vertex-colouring is obtained, it can be used to implement the above strategies by associating a colour with a particular sub-carrier frequency band or a particular time slot as required. These assignments could be done during the time of network deployment with an associated one-time cost.

When links between adjacent nodes are not ideal noiseless links, the broadcast information received could be distorted. Knowledge of the channel state information (CSI) could be used to further regulate the forwarding probability at every node to overcome channel outages. Each link could be modelled to be either in a 'good' or a 'bad' state based on CSI statistics. The problem then reduces to carrying out the probabilistic forwarding mechanism on a random subgraph of the original network. The techniques necessary to analyze this scenario are more sophisticated than those used in this thesis, and form a possible future research direction.

For the ease of analysis, in this thesis, we think of transmissions to be happening one after the other in the network. In other words, at any time, one node in the network transmits a single packet. We can make this assumption since our analysis does not account for the time delay for a packet to traverse from the source to any other network node. Moreover, with this assumption, packet collisions are avoided. It is to be highlighted that this is only for convenience in our analysis, and any practical implementation of the algorithm will require an appropriate scheduling scheme as discussed before.

Chapter 3

Simulations

The purpose of this chapter is two-fold. Firstly, it serves as a platform to introduce and formally define the network topologies that we will discuss as part of this thesis. Secondly, simulations on these different underlying network topologies provides an initial understanding of the performance of the probabilistic forwarding mechanism with coded packets.

The required scripts for carrying out the simulations of this chapter are made available through a Github repository [72].

3.1 Random geometric graphs

Our main interest is to characterize the probabilistic forwarding mechanism on random geometric graphs since these are used widely to model ad-hoc networks. We start by giving a procedure to construct them.

A random geometric graph (RGG), G_m, on a finite area $\Gamma_m := \left[\frac{-m}{2}, \frac{m}{2}\right]^2 \subset \mathbb{R}^2$ is defined using two parameters: the intensity λ and the distance threshold r. It is constructed as follows:

- **Step 1:** Sample the number of points, N, from a Poisson distribution with mean $\lambda\nu(\Gamma_m)$. Here, $\nu(\cdot)$ is the Lebesgue measure on $\mathbb{R}^2$. Therefore, $N \sim \text{Poi}(\lambda m^2)$.

- **Step 2:** Choose points $X_1, X_2, \cdots, X_N$ uniformly and independently from Γ_m.

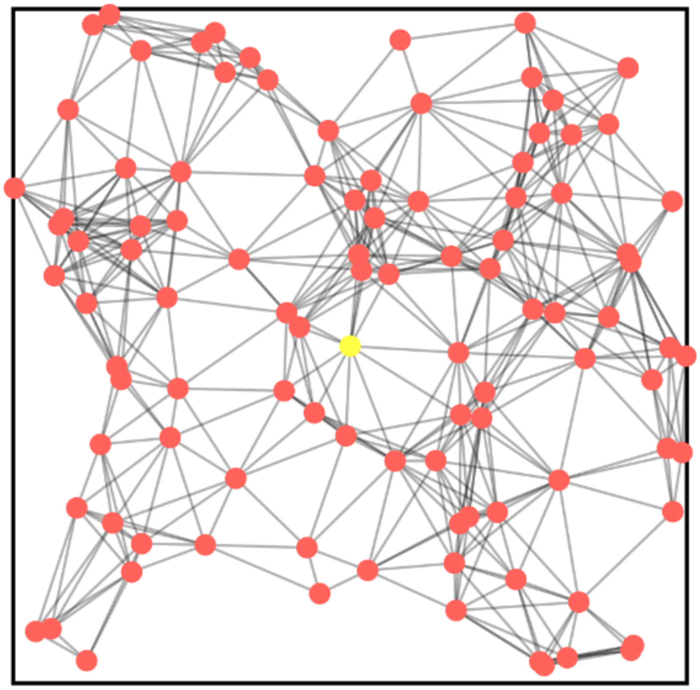


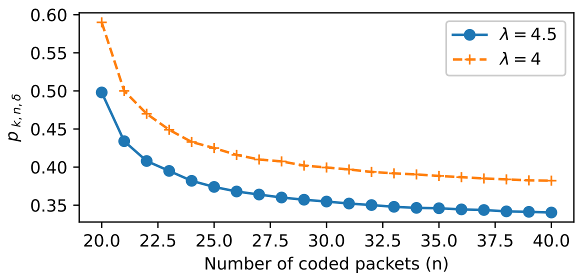
Figure 3.1: A random geometric graph (RGG) with $\lambda = 4$ and $r = 1$ on Γ_5.

These form the points of a Poisson point process (see [73, Section 2.5]) Φ, and constitute the vertex set of G_m.

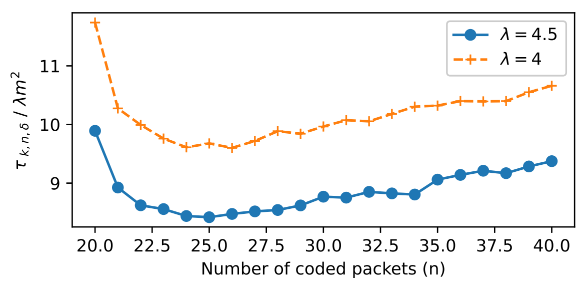
- **Step 3:** Place an edge between any two vertices which are within Euclidean distance r of each other.

It suffices to study RGGs by keeping one of the parameters fixed. In our treatment, we will fix the distance parameter r to be equal to 1, and study various properties as a function of the intensity, λ. Notice that the total number of nodes in the network is a random number given by $N = \Phi(\Gamma_m)$. An illustration of an RGG with $\lambda = 4$ and $r = 1$ is provided in Fig. 3.1.

We will assume that a node is present at the origin $\mathbf{0} = (0,0) \in \mathbb{R}^2$ which acts as the source and initiates the broadcast. Since we are interested in a broadcast problem, we limit our network to the nodes which are present within the component of the origin and the edges connecting them. Call this graph $G_m^{\mathbf{0}}$. The nodes within this component are the nodes which are *reachable* from the source.



(a) Minimum retransmission probability



(b) Expected total number of transmissions

Figure 3.2: Simulations on a random geometric graph generated on Γ_{101} with intensity λ and distance threshold $r = 1$. Probabilistic forwarding done with $k = 20$ packets and $\delta = 0.1$

Simulations were performed on two RGGs, generated on Γ_{101} with intensity $\lambda = 4.5$ and 4. As stated before, the distance threshold parameter r was set to 1. The probabilistic forwarding mechanism was carried out with $k = 20$ packets and n varying from 20 to 40. The value of δ was set to 0.1. Twenty realizations of $G_m^{\mathbf{0}}$ were generated and 10 iterations of the probabilistic forwarding mechanism was carried out on each of the realizations. The fraction of successful receivers was averaged over each iteration and realization of the graph. This was used to find the minimum forwarding probability, $p_{k,n,\delta}$, required for a near-broadcast, which is plotted in Fig. 3.2(a). The $p_{k,n,\delta}$ values so obtained were further used to find the expected total number of transmissions over the same realizations. The expected total number of transmissions $\tau_{k,n,\delta}$, normalized by the number of points in Γ_m, is shown in Figure 3.2(b).

Notice that the expected number of transmissions decreases initially to a minimum and then increases. The point with $n = k = 20$ in both the plots corresponds to the probabilistic forwarding mechanism with no coding. The initial decrease, till around $n = 25$, indicates the benefit of introducing coding along with probabilistic forwarding. The number of coded packets, n, and the probability, $p_{k,n,\delta}$, corresponding to the minimum point of Figure 3.2(b) are the ideal parameters for operating the network to obtain maximum energy benefits while ensuring a near-broadcast.

3.2 Lattices

The advantage brought about by the introduction of coded packets with probabilistic forwarding is more clearly seen in lattice structures. Here, we present simulations on the square grid, the triangular grid and the $3D$ cubic lattice.

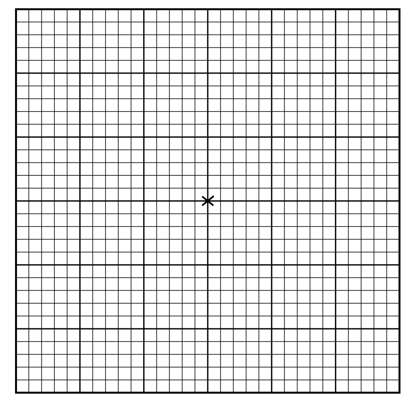
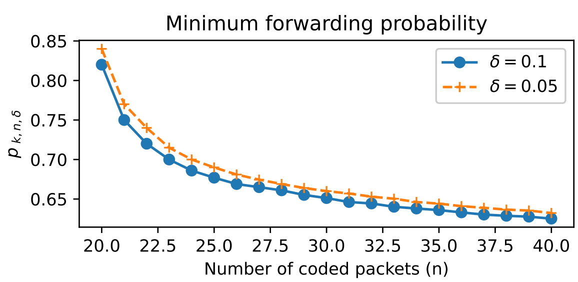


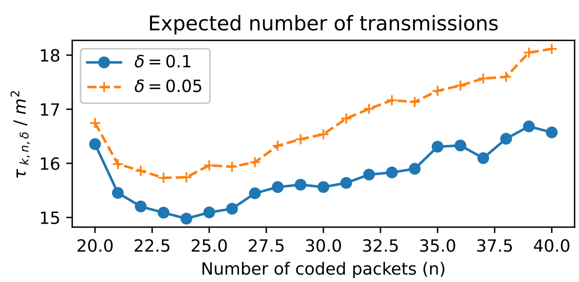
Figure 3.3: 31×31 grid.

The vertex set of an $m \times m$ *square grid* for odd m is given by $\left[-\frac{(m-1)}{2}, \frac{(m-1)}{2}\right]^2 \cap \mathbb{Z}^2 = \Lambda_m \cap \mathbb{Z}^2$. The edge set is $\{(i,j) \mid |i-j| = 1 \text{ where } i,j \in \Lambda_m \cap \mathbb{Z}^2\}$. Here, $|\cdot|$ indicates the ℓ_2 norm. An illustration of a 31×31 square grid is shown in Fig. 3.3. The source is present at the origin $\mathbf{0} = (0,0) \in \mathbb{Z}^2$ which is the center of the grid. The total number of nodes on the $m \times m$ square grid is $N = m^2$.

Simulations on the 31×31 square grid are shown in Fig. 3.4. The probabilistic mechanism was carried out with $k = k_s = 20$ packets and $\delta = 0.1$ and 0.05 and n varying from 20 to 40 packets. As in the case of a RGG, the expected total number of



(a) Minimum retransmission probability



(b) Expected total number of transmissions

Figure 3.4: Simulations on a 31×31 grid. Probabilistic forwarding done with $k = 20$ packets.

transmissions initially decreases to a minimum and then gradually increases indicating an energy advantage associated with the addition of coded packets.

The *triangular grid* can be thought of as the graph with the vertex set being the same as the square grid, $\Lambda_m \cap \mathbb{Z}^2$, but with additional edges between vertices $i = (i_x, i_y)$ and $j = (i_x + 1, i_y + 1)$ for $i, j \in \Lambda_m \cap \mathbb{Z}^2$. An illustration of the triangular grid is provided in Fig. 3.5. Since there are far more edges on the triangular grid as compared to the square grid, there are more paths from the origin to any node in the network for a packet to propagate. It is due to this reason that the forwarding probability required for a near-broadcast on a triangular grid is lesser compared to that on a square grid of the same dimensions as seen in Fig. 3.4(a) and 3.6(a).

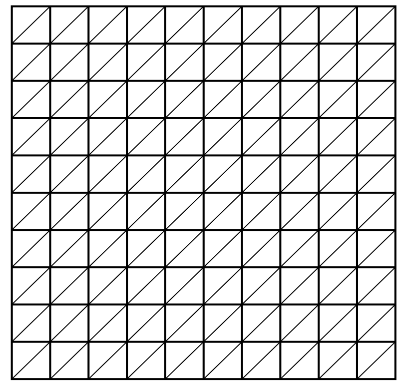
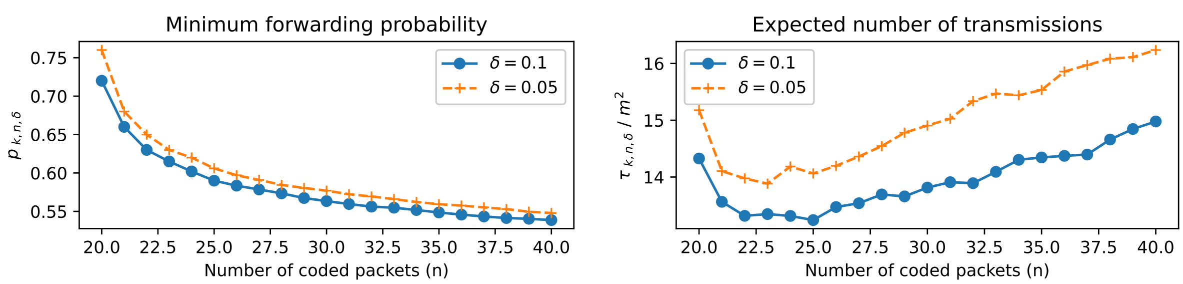


Figure 3.5: Triangular grid.



(a) Minimum retransmission probability

(b) Expected total number of transmissions

Figure 3.6: Simulations on a 31×31 triangular grid. Probabilistic forwarding done with $k = 20$ packets.

3.3 Trees

The advantage seen with respect to the expected total number of transmissions due to the introduction of coded packets with probabilistic forwarding on RGGs and lattice structures, is not replicated on tree-like structures. In this section, we consider rooted trees in which the root is designated as the source. We provide simulation results which

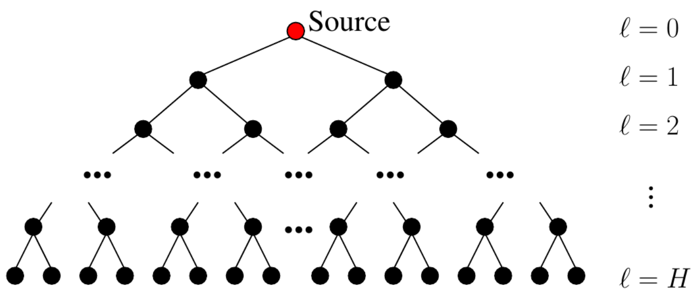
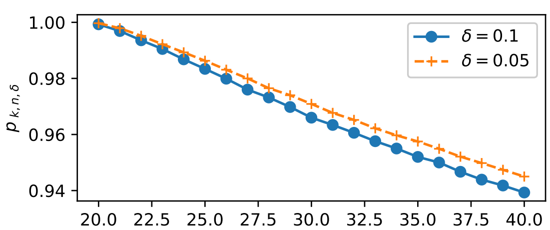


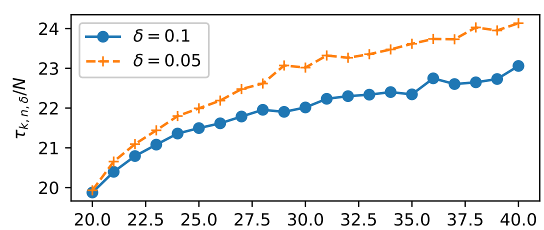
Figure 3.7: A rooted binary tree of height H.

convey a very different story from that of the previous two sections.

A rooted binary tree of height H is the graph depicted in Fig. 5.1. The tree consists of H levels, with the root node at level $\ell = 0$, and for $\ell = 1, 2, \ldots, H-1$, each node at level ℓ having two children at level $\ell+1$. Thus, there are 2^ℓ nodes at level ℓ, for $\ell = 0, 1, 2, \ldots, H$, so that the total number of nodes in the tree is $N = \sum_{\ell=0}^{H} 2^\ell = 2^{H+1} - 1$. The root node is taken to be the source node.



(a) Minimum retransmission probability



(b) Expected total number of transmissions

Figure 3.8: Simulations on a binary tree of height $H = 10$. Probabilistic forwarding done with $k = 20$ packets.

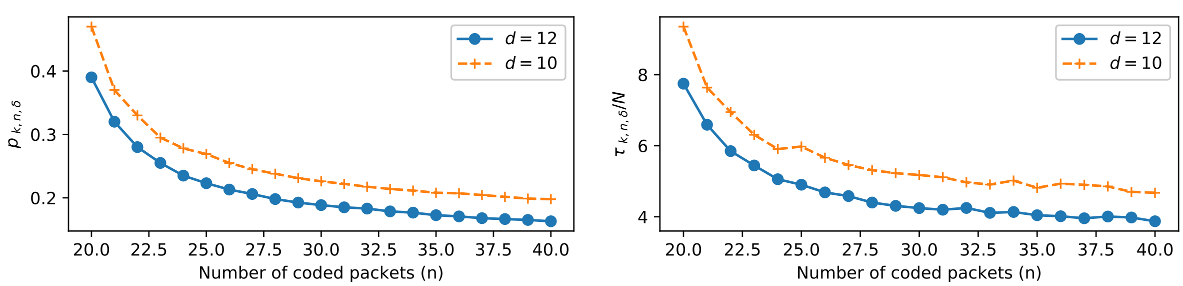
Simulations of the probabilistic forwarding mechanism with $k = 20$ message packets encoded into n coded packets (n varying from 20 to 40) on a rooted binary tree of height 10 are shown in Fig. 3.8. Notice that the minimum forwarding probability necessary for a near-broadcast decreases with increase in n as in the other graphs. However, the expected total number of transmissions, $\tau_{k,n,\delta}$, does not show any decrease. It increases with n. This means that even though there are additional coded packets which can assist in a near-broadcast, transmitting them results in a larger number of transmissions. Therefore, coding along with probabilistic forwarding on trees does not help to save energy.

3.4 Other graphs

3.4.1 Hypercube

The d-dimensional hypercube has the vertex set $V = \{0,1\}^d$ with edges $(x, y) \in E$ if x and y represented as binary strings differ in exactly one bit. The number of vertices is $N = 2^d$. Since each vertex has degree d, the number of edges is $d2^{d-1}$.

Simulation of the probabilistic forwarding protocol with coded packets was carried out on a d-dimensional hypercube with $k = 20$ packets and n varying from 20 to 40. The all zeros vertex, $\mathbf{0} = (0, 0, \cdots, 0) \in \{0,1\}^d$ was taken to be the source. The plots for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$ are shown in Fig. 3.9. The plot for the expected number of transmissions seems



(a) Minimum retransmission probability

(b) Expected total number of transmissions

Figure 3.9: Simulations on a hypercube with 2^d nodes. Probabilistic forwarding done with $k = 20$ packets.

to decrease monotonically for the range of n shown in the figure. Note the very low values of the minimum forwarding probability required for a near-broadcast. For example, a forwarding probability of 0.2 suffices to ensure a near-broadcast when $k = 20$ message packets are encoded into $n = 30$ coded packets. The number of transmissions at this probability is approximately 4.4 on an average, per node. This is much smaller than the number of data packets, $k = 20$, in the network.

Additionally, suppose that every node in the network possesses all the 30 packets, and transmits each with probability $p_{k,n,\delta} = 0.2$. Then, on an average, there would be $\tau_{k,n,\delta} = 0.2 \times 30 = 6$ transmissions per node. Similarly, if every node in the network (ignore the source and its neighbours) received exactly 20 packets and forwarded each of them probabilistically, the average transmissions per node would be $\tau_{k,n,\delta} = 0.2 \times 20 = 4$. However, notice that the number of transmissions per node obtained through simulations, which is 4.4, is in between these two values. This indicates that nodes in the network, on an average, receive around 22 packets. This is greater than the minimum number of packets ($k = 20$) required to recover the k_s message packets from the source.

3.4.2 Random regular graphs

In this subsection, we consider a class of random graphs called random regular graphs (RRGs). Fix a pair of positive integers $1 \leq d < N$ such that Nd is even, and let $S(N, d)$ be the set of all simple[1] d-regular graphs on a set of N vertices, V_N. The uniform random d-regular graph $G_{N,d}$ is obtained by sampling with respect to the uniform distribution on $S(N, d)$. The probability of each simple d-regular graph is given by $\frac{1}{|S(N,d)|}$.

One of the popular ways of generating random regular graphs is via the *configuration model*, proposed by Bollobás in [74]. Here, we first create a set of points $\mathbf{P} = \{1 \times [d], 2 \times [d], \cdots, N \times [d]\}$, where $[d] = \{1, 2, \cdots, d\}$. This set contains d points corresponding to each of the N vertices of the graph. Clearly there are Nd points on the whole. Each element of $\mathbf{P}$ represents a half-edge (or a stub) emanating from a vertex. Let M be a uniformly random perfect matching of the points in $\mathbf{P}$ [2]. It should be noted that Nd must be even for a perfect matching to exist. Since $\mathbf{P}$ has Nd elements, there are $\frac{(Nd)!}{(Nd/2)!\ 2^{Nd/2}}$ perfect matchings on it. We can obtain a (multi)graph, $GM(d)$, if we project $\mathbf{P}$ onto V_N, preserving adjacencies, i.e., for any two vertices $i, j \in V_N$, if M contains an edge between a point in $i \times [d]$ and a point in $j \times [d]$, then $GM(d)$ contains the edge (i, j). Notice that

[1] A simple graph has no loops or multiple edges.

[2] In a perfect matching of a set A with $|A|$ being even, elements of A are divided into pairs such that every element of A is present in exactly one pair. A uniformly random perfect matching is obtained by sampling from the set of all perfect matchings, uniformly at random.

this may not result in a simple graph. However, if we condition on the event that $GM(d)$ is a simple graph, then it is uniformly distributed over the set of simple d-regular graphs. This is because any permutation of the d half-edges at every vertex forming the simple graph, also results in the same graph. In other words, there are $(d!)^N$ matchings which give rise to the same simple graph. Since this is true for every simple graph generated using the configuration model, we obtain a uniform distribution upon conditioning on the graph $GM(d)$ being simple. We take this to be the graph $G_{N,d}$.

The procedure outlined above can be used to generate random graphs with a given degree sequence $\mathbf{d} = (d_1, d_2, \cdots, d_N)$, where d_i is the degree of vertex i. The matching is then constructed on $\mathbf{P} = \{1 \times [d_1], \cdots, N \times [d_N]\}$. We will use this in our analysis of the probabilistic forwarding mechanism on RRGs.

Since our interest is only in graphs with no loops or multiple edges, we have to generate these perfect matchings until the time we obtain a simple graph. This procedure is time-consuming. To overcome this, we follow an adaptive approach proposed in [75] to generate the graph $G_{N,d}$ which is detailed in Algorithm 1.

In [76], the authors show that the graph generated using the procedure of Algorithm 1 has a uniform distribution asymptotically in N as long as $d < N^{\frac{1}{3}}$. This provides a fast method to generate asymptotically uniform random regular graphs with degree up to $N^{\frac{1}{3}} - \varepsilon$, for any positive constant $\varepsilon > 0$. We run the probabilistic forwarding algorithm on this graph $G_{N,d}$. The source is chosen uniformly at random from the N vertices of the graph. It is known that the graph $G_{N,d}$ is almost surely connected. In fact, in [77], it is shown that $G_{N,d}$ almost surely has vertex-connectivity[3] d, for $d \geq 3$.

Simulations are carried out on two random d-regular graphs for $d = 4$ and $d = 8$. Twenty realizations of each graph are generated and 10 iterations of the probabilistic forwarding protocol are carried out on each. The results of the simulation are plotted in Fig. 3.10. The plots show similar trends as those on the hypercube. The minimum forwarding probability required for a near-broadcast and the expected number of transmissions decreases as the number of coded packets is increased. The RRG with degree

[3] It is the minimum number of vertices whose deletion causes the graph to be disconnected.

Algorithm 1: Generate random d-regular graph on N vertices

Result: Output a random d-regular graph $G_{N,d}$

Let $V_N = \{1, 2, \cdots, N\}$ and the edge set $E_N = \emptyset$.

Let $\mathbf{U} = \{1 \times [d], 2 \times [d], \cdots, N \times [d]\}$ denote the set of unpaired points.

Define a pair (i, a_i) and (j, a_j) in U to be *suitable* if addition of the edge (i, j) to E_N does not create any loop or a multiple edge.

while *Suitable pair can be found* **do**

  Choose two random points (i, a_i) and (j, a_j) in $\mathbf{U}$;

  if *suitable* **then**

    Include the edge (i, j) in E_N;

    Delete (i, a_i) and (j, a_j) from U;

  end

end

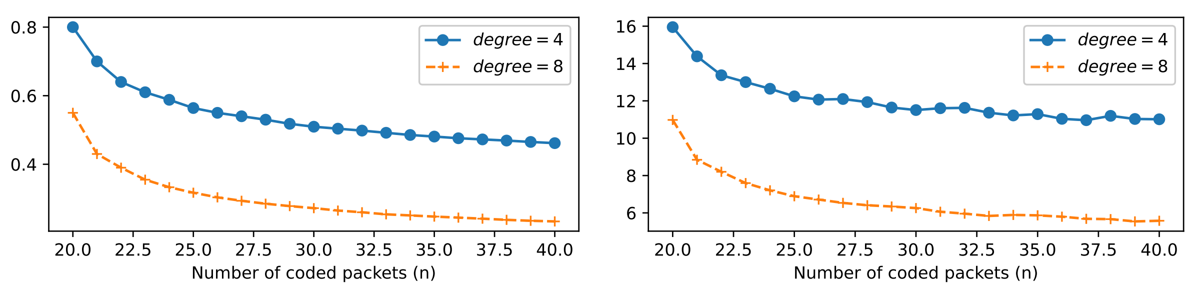
if $G_{N,d} = (V_N, E_N)$ *is d-regular* **then**

  Output $G_{N,d}$;

else

  Repeat Algorithm 1 ;

end



(a) Minimum retransmission probability

(b) Expected total number of transmissions

Figure 3.10: Simulations on a random d-regular graph with 1000 nodes. Probabilistic forwarding done with $k = 20$ packets.

$d = 8$ is better connected, and hence requires fewer nodes to transmit packets to achieve a near-broadcast, as compared to the RRG with $d = 4$. This in turn brings down the value of $p_{k,n,\delta}$ required for a near-broadcast. This is seen from the plot as well.

3.5 Takeaways

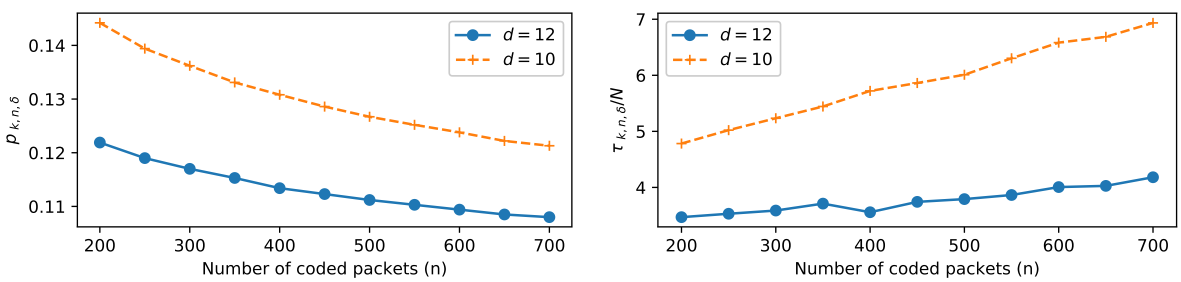
In this section, we put down some observations that we can make from the simulation results.

- The minimum forwarding probability decreases with the introduction of additional coded packets in the network. This is natural to expect since, with extra packets, one can afford to transmit each packet at a lower probability while still ensuring that there is a near-broadcast. In fact, in Chapter 4, we will see that $p_{k,n,\delta}$ indeed diminishes to 0 as $n \to \infty$, for any underlying network topology.

- On well-connected graphs such as grids, RGGs and lattice structures, the expected total number of transmissions, $\tau_{k,n,\delta}$, decreases to a minimum and then gradually increases. The value of the number of coded packets, n, and the value of the forwarding probability, $p_{k,n,\delta}$, corresponding to this minimum are optimal in terms of the energy expenditure for a near-broadcast. More specifically, the network when operated at these parameters has minimal expected number of transmissions while ensuring a near-broadcast.

- On trees, the expected total number of transmissions does not decrease with the addition of coded packets. As a matter of fact, we will show in Chapter 5 that it actually increases. This implies that introduction of coded packets along with the probabilistic forwarding protocol degrades the performance, since there are unnecessary transmissions of the additional coded packets.

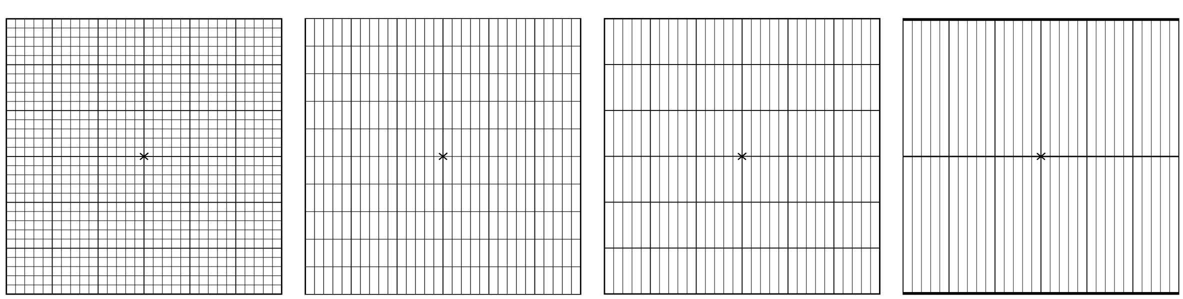
- On hypercubes and random regular graphs, the plot for the expected number of transmissions gives an impression that it decreases gradually on increasing the number of coded packets, n. However, note that, since the source always transmits each of the n packets, the expected number of transmissions should start to increase after some (possibly large) n. To illustrate this, in Fig. 4.2, we provide simulations on the hypercube for a wider range of n. Specifically, we vary n from 200 to 700 in steps of 50 and plot the corresponding values of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$.



(a) Minimum retransmission probability (b) Expected total number of transmissions

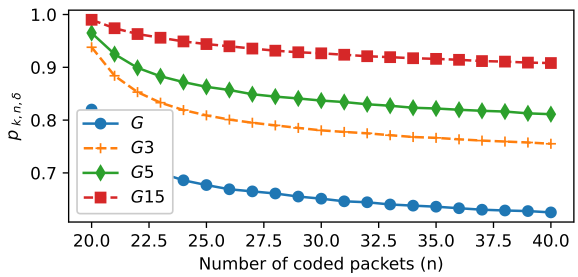
Figure 3.11: Simulations on a d-dimensional hypercube. Probabilistic forwarding done with $k = 20$ packets.

We will see in Chapter 5 that the primary reason for not observing any benefit (with respect to the reduction in the overall number of transmissions) on trees is due to the presence of a unique path from the source to any other node on the tree. On graphs such as lattices, this phenomenon seems to arise from the availability of "multipath diversity" in the network, i.e., the existence of multiple paths between the source node and any

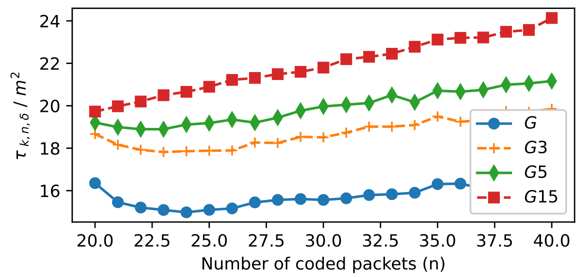


(a) 31×31 grid (G) (b) Every third row ($G3$) (c) Every fifth row ($G5$) (d) Boundary and the center row ($G15$)

Figure 3.12: Graphs to illustrate the importance of multiple paths.



(a) Minimum retransmission probability



(b) Expected total number of transmissions

Figure 3.13: Comparison of simulations on a grid of side length $m = 31$ (G), retaining every third ($G3$), every fifth ($G5$) and every fifteenth ($G15$) row of edges. Probabilistic forwarding done with $k = 20$ packets.

other node in the network. Indeed, in a binary tree, there is only one path from the root to any other node, whereas in a large grid, there is abundant multipath diversity.

To test our multipath diversity hypothesis more systematically, we performed further simulations of the probabilistic forwarding protocol on graphs with different levels of multipath diversity. Starting with the 31×31 grid G depicted in Fig. 3.12(a), we systematically deleted edges to obtain subgraphs $G3$, $G5$ and $G15$ with lower multipath diversity. Specifically, the graph Gq (for $q = 3, 5, 15$) was obtained from the grid G as follows. The nodes of G form a 31×31 array, whose rows can be indexed by the integers $0, 1, 2, \ldots, 30$, with 0 denoting the index of the topmost row. Then, Gq is obtained from G by retaining the horizontal edges connecting adjacent nodes in row j, for every j that is a multiple of q, and deleting all other horizontal edges — see Figs. 3.12(b)–(d). The multipath diversity evidently decreases as q increases and the graph obtained, Gq, becomes more tree-like. The results of our simulations, for $k = 20$ packets, with the expected fraction of nodes receiving at least k packets being $1 - \delta = 0.9$, are shown in Fig. 3.13. In these simulations, the source node is the node at the centre of the grid, depicted by a '$\times$' in each of the graphs in Fig. 3.12.

Chapter 4

Initial observations

In this chapter, we formalize some notions observed in the simulations. We first begin by comparing the number of transmissions in our mechanism with those of classic algorithms like flooding and probabilistic forwarding with no coding. We then proceed to justify some of the observations that were made in the previous chapter regarding the behaviour of the minimum forwarding probability, $p_{k,n,\delta}$, and the expected total number of transmissions, $\tau_{k,n,\delta}$.

4.1 Comparison of broadcast schemes

Flooding as a broadcast mechanism involves a node forwarding every newly received packet to all its one-hop neighbours. If there are N nodes in the network and the source has k_s message packets, then the total number of transmissions is k_sN. However, a node might receive the same packet from multiple neighbours resulting in wasteful transmissions.

In probabilistic forwarding, every node in the network forwards a newly received packet with probability p and does nothing with probability $1-p$. An upper bound on the expected number of transmissions for this algorithm can be obtained thus. On a network of N nodes, an average of Np nodes decide to transmit a given source packet, irrespective of whether they receive it or not. Since there are k_s source packets in all, there can be at most k_sNp expected total number of transmissions. Note that $p = 1$ corresponds to the flooding protocol. Thus, with a forwarding probability $p < 1$, there are gains to be had over flooding.

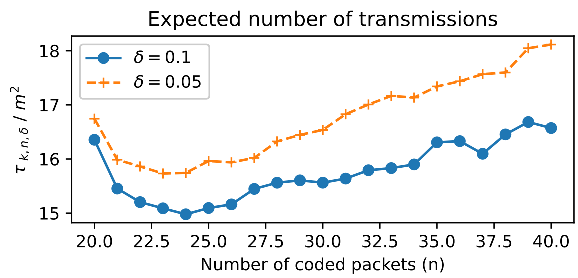


Figure 4.1: Simulations on a 31×31 grid. Probabilistic forwarding done with $k = 20$ packets.

Introduction of coded packets along with probabilistic forwarding brings down the expected number of transmissions even further on some network topologies like grids, hypercubes, lattice structures etc. An exploration of this phenomenon via simulations was presented in Chapter 3. Simulation results in Figs. 3.2, 3.4, 3.9 etc. indicate that the total number of transmissions further decreases, as compared to probabilistic forwarding with no coding. Indeed, this is seen on most well-connected graphs for a limited range of values of the number of coded packets n. For example, consider the simulation plot of the number of transmissions on a square grid replicated here from the previous chapter in Fig. 4.1. The point $n = k = 20$ corresponds to probabilistic forwarding with no coding. Notice the decrease in the number of transmissions when n is increased from 20 to 25. While the exact decrease is hard to quantify, on random geometric graphs (Chapter 7) and grids (Chapter 6), we use ergodic theorems to obtain estimates and thus justify the benefit obtained with the introduction of coded packets along with probabilistic forwarding.

4.2 Running time

The proposed probabilistic forwarding mechanism with coded packets runs in finite time on a given network. This is easily seen since the decision to either transmit a particular packet or not is made only once. Subsequent receptions of the same packet are neglected.

Assuming packets are transmitted sequentially with only a single network node transmitting at any point of time, a naive bound on the mean running time of the algorithm is the expected total number of transmissions for n packets which is easily upper bounded by nNp.

4.3 Behaviour of $p_{k,n,\delta}$

Recall the definition of the minimum forwarding probability:

$$p_{k,n,\delta} \ := \ \inf \left\{ p \;\middle|\; \mathbb{E}\left[\frac{R_{k,n}}{N}\right] \geq 1-\delta \right\},$$

where $R_{k,n}$ is the number of nodes which receive at least k out of n packets (successful receivers). For a deterministic graph, the expectation is over the probabilistic forwarding algorithm, whereas, for a random graph, it is additionally over all the realizations of the graph. Additionally, recall that for RGGs and RRGs, the total number of nodes, N, is the number of nodes in the component containing the source, which is also a random quantity.

If a successful receiver must receive k out of n' coded packets, instead of k out of n, where $n' > n$, each packet can be transmitted at a lower probability while still ensuring a near-broadcast. In fact, the minimum forwarding probability tends to 0 as n is increased. This is formalized in the following lemma.

Lemma 4.3.1. *Consider probabilistic forwarding of n coded packets on a fixed underlying connected graph. For fixed values of k and δ,*

(a) $p_{k,n,\delta}$ *is a non-increasing function of n.*

(b) $p_{k,n,\delta} \to 0$ *as* $n \to \infty$.

Proof. (a) For any $n > 0$, the random variables $R_{k,n}$ and $R_{k,n-1}$ can be coupled as follows: If there are a total of n coded packets, then $R_{k,n-1}$ (resp. $R_{k,n}$) is realized as the number of nodes, including the source node, that receive at least k of the *first* $n-1$ (resp. at least

k of the n) coded packets. It is then clear that $\mathbb{E}[\frac{1}{N}R_{k,n}] \geq \mathbb{E}[\frac{1}{N}R_{k,n-1}]$, and hence, by (2.1), we have $p_{k,n,\delta} \leq p_{k,n-1,\delta}$.

(b) From the n coded packets, create $\lfloor \frac{n}{k} \rfloor$ non-overlapping (i.e., disjoint) groups of k packets each. For $i = 1, 2, \cdots, \lfloor \frac{n}{k} \rfloor$, let A_i be the event that the ith group of k coded packets is received by at least $(1-\delta/2)N$ nodes. The events A_i are mutually independent and have the same probability of occurrence. For any $p > 0$, we have $\mathbb{P}(A_i)$ being strictly positive (but perhaps small). Hence,

$$\mathbb{P}(\text{at least one } A_i \text{ occurs}) = 1 - \big(1 - \mathbb{P}(A_1)\big)^{\lfloor \frac{n}{k} \rfloor} \geq 1 - \frac{\delta}{2}$$

for all sufficiently large n, so that $\mathbb{P}\left(\frac{R_{k,n}}{N} \geq 1 - \delta/2\right) \geq 1 - \delta/2$. This further implies that $\frac{\mathbb{E}\left[R_{k,n}\right]}{N} \geq (1-\delta/2)(1-\delta/2) \geq 1-\delta$. Thus, for any $p > 0$, we have $p_{k,n,\delta} \leq p$ for all sufficiently large n. □

For the case of RGGs, Lemma 4.3.1 holds for every realization of the random graph. To make a formal statement, let us denote the RGG by G, and a realization of it by g. Since N is the number of nodes in the component of the source, it is a random quantity. The successful receivers are a subset of the nodes in the component of the origin. Let us define $\mathbb{E}_g$ to be the expectation over the probabilistic forwarding protocol when the underlying graph is g. Using the tower property of expectation, we obtain

$$\mathbb{E}\left[\frac{R_{k,n}}{N}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{R_{k,n}}{N} \,\middle|\, G\right]\right].$$

Conditioned on a realization g of G, N is fixed and it is true that $\mathbb{E}_g\left[\frac{R_{k,n}}{N}\right] \geq \mathbb{E}_g\left[\frac{R_{k,n-1}}{N}\right]$ due to similar arguments as in Lemma 4.3.1(a). Therefore, we have that $p_{k,n,\delta}$ is a non-increasing function of n even when the underlying graph is random.

In a similar way, due to Lemma 4.3.1(b), for all sufficiently large n, we have that $\mathbb{E}_g\left[\frac{R_{k,n}}{N}\right] \geq 1-\delta$ for any realization g of G. This holds true since the events A_i defined in the proof of Lemma 4.3.1(b) are independent and identically distributed (iid) conditional on $G = g$. Therefore $\mathbb{E}\left[\frac{R_{k,n}}{N}\right]$ can be made arbitrarily close to 1 for sufficiently large n.

This in turn means that $p_{k,n,\delta} \to 0$ as $n \to \infty$.

These arguments can be extended to the case of an RRG as well by conditioning both on the realization of the graph and the choice of the source vertex.

4.4 Behaviour of $\tau_{k,n,\delta}$

A much more interesting trend is that of the expected total number of transmissions $\tau_{k,n,\delta}$, which initially decreases and then grows gradually as the number of coded packets n is increased. There is thus an optimal value of n that minimizes $\tau_{k,n,\delta}$. This happens due to an interplay between two opposing factors: an increase in n leads to a decrease in $p_{k,n,\delta}$, which contributes towards a decrease in $\tau_{k,n,\delta}$. But this is opposed by the fact that a higher redundancy tends to increase the number of transmissions, since there are a larger number of packets to be transmitted in the network. More formally, the expected total number of transmissions can be expressed as

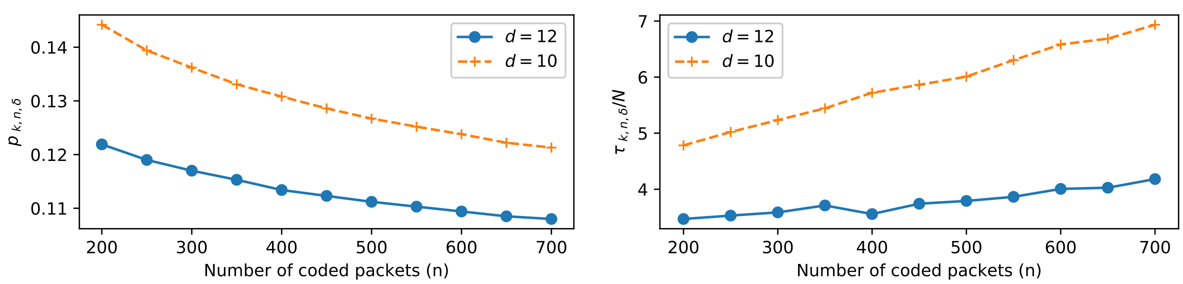
$$\tau_{k,n,\delta} = \sum_{i=0}^{n} \mathbb{E}\left[T_i\right],$$

where T_i is the number of transmissions of packet i. Note that with the addition of coded packets in the network, the number of terms of this summation increases. However, from Lemma 4.3.1, the probability with which each packet is forwarded decreases with the addition of excess coded packets. This in turn reduces the individual terms of the summation which is the number of transmissions of each packet, T_i.

On well-connected graphs such as the lattice structures, the initial decrease in $\tau_{k,n,\delta}$ can be attributed to the dominant effect of the initial steep decrease in $p_{k,n,\delta}$. However, as the redundancy is further increased, the decrease in $p_{k,n,\delta}$ becomes more gradual. In this regime, as the number of coded packets n increases, the gain obtained via the slight decrease in $p_{k,n,\delta}$ is more than offset by the fact that there are more packets to be transmitted in the network.

On a tree, however, the decrease in the forwarding probability $p_{k,n,\delta}$ is not substantial. It can be observed from Fig. 3.8(a) that the minimum forwarding probability necessary for a near-broadcast remains very close to 1, even with the introduction of additional coded packets. This means that even though there are additional packets in the system, each of them is being transmitted with probability ≈ 1. This only contributes to an increase in the number of transmissions in the network which is what is observed on trees in Fig. 3.8(b).

Additionally, note that a very naive bound for the expected total number of transmissions is $\tau_{k,n,\delta} > n$. This is because the source always transmits each of the n packets. This means that, after a certain n, the curve for $\tau_{k,n,\delta}$ must eventually stop decreasing in n. Simulations on the hypercubes and RRGs in Figs. 3.9 and 3.10 showed a monotonic decrease in $\tau_{k,n,\delta}$ for n ranging from 20 to 40. However, for higher values of n, this can indeed be seen to increase with n as shown in Fig. 4.2 for the hypercube. At these ranges of n, the minuscule decrease in $p_{k,n,\delta}$ is unable to compensate for the increase in the number of transmissions of each packet.



(a) Minimum retransmission probability

(b) Expected total number of transmissions

Figure 4.2: Simulations on a hypercube with 2^d nodes. Probabilistic forwarding done with $k = 20$ packets.

Part II

Deterministic graphs

Chapter 5

Trees

In this chapter, we analyze the probabilistic forwarding mechanism on trees. We consider rooted trees in which the root is designated as the source. We first analyze the mechanism on rooted binary trees and then generalize it to d-ary trees, spherically symmetric trees and some other general tree configurations.

5.1 Binary tree

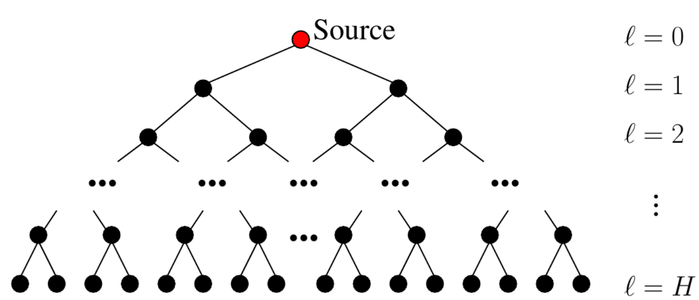
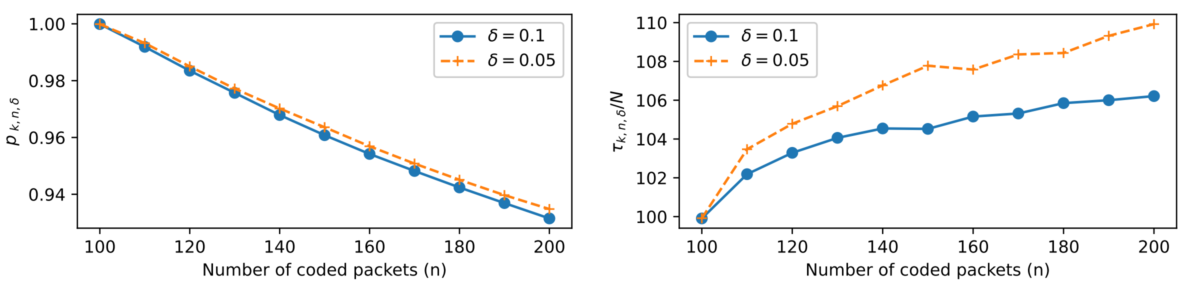


Figure 5.1: A rooted binary tree of height H.

Recall the rooted binary tree of height H defined in Section 3 and depicted here in Fig. 5.1. There are totally $N = \sum_{l=0}^{H} 2^l = 2^{H+1} - 1$ nodes and the root node is taken to be the source. The root node encodes the k data packets into n coded packets and transmits them to its children. Every other node on the tree follows the probabilistic forwarding strategy with some fixed forwarding probability $p > 0$.

Simulation results of the probabilistic forwarding mechanism on a binary tree of height $H = 10$ with $k = 100$ packets are shown in Fig. 5.2. Our goal is to explain these results which reflect the behaviour of the probabilistic forwarding protocol on trees. We first begin by giving an intuitive explanation for this behaviour.



(a) Minimum retransmission probability (b) Expected total number of transmissions

Figure 5.2: Simulations on a binary tree of height $H = 10$. Probabilistic forwarding done with $k = 100$ packets.

On the binary tree, almost half the number of nodes are present on the leaves of the tree. Since we are interested in a broadcast mechanism where the fraction of successful receivers is at least $1 - \delta$ for $\delta = 0.1$(say), we would want a large fraction (0.8 to be precise) of the leaves to receive at least k out of the n packets. The probability that a leaf node receives a packet is p^{H-1} since, every node on the unique path connecting the leaf node to the source needs to transmit the packet. If n packets are transmitted, a leaf node receives np^{H-1} packets on average. Since the leaf node needs to receive at least k packets for it to be successful, we must have $np_{k,n,\delta}^{H-1} \geq k$. Notice that since $\frac{k}{n} \geq \frac{1}{2}$, for large H, the minimum forwarding probability for a near-broadcast must be close to 1. This offsets the advantage seen due to the decrease in the forwarding probability and hence the number of transmissions increases with n. We formalize this intuition in the next section.

5.2 Main results

To get a handle on the minimum retransmission probability $p_{k,n,\delta}$ for a near-broadcast, we first look at the number of successful receivers, $R_{k,n}$. We can write $R_{k,n} = \sum_{\ell=0}^{H} R_\ell$, where R_ℓ is the number of nodes at level ℓ that hold at least k of the n packets. Similarly, define $T_{k,n} = \sum_{\ell=0}^{H} T_\ell$, where T_ℓ is the number of transmissions by nodes at level ℓ. Note that $T_0 = n$ and $R_0 = 1$ since the source transmits all the n packets.

In a tree, there is only a single path from the root to any node in the tree. Thus, for a node $\mathbf{v}$ at level ℓ to receive a packet from the root, all the intermediate nodes on the unique path from the root to $\mathbf{v}$ need to transmit the packet. Hence, for $\ell \geq 1$,

$$\mathbb{P}(\text{node } \mathbf{v} \text{ at level } \ell \text{ receives the } j\text{th packet}) = p^{\ell-1}.$$

Since individual packets are transmitted independently of each other, we have

$$\begin{aligned}\mathbb{P}(\text{node } \mathbf{v} \text{ at level } \ell \text{ receives at least } k \text{ out of } n \text{ packets}) &= \sum_{r=k}^{n} \binom{n}{r} p^{(\ell-1)r}(1-p^{\ell-1})^{n-r} \\ &= \mathbb{P}(Z_{\ell-1} \geq k),\end{aligned}$$

where $Z_{\ell-1} \sim \mathrm{Bin}(n, p^{\ell-1})$ is a binomial random variable with parameters n and $p^{\ell-1}$. Nodes that share a common parent receive the same packets and hence will possess the same number of packets at the end of the probabilistic forwarding mechanism. Summing the above over all nodes $\mathbf{v}$ at level ℓ, we obtain $\mathbb{E}[R_\ell] = 2^\ell\, \mathbb{P}(Z_{\ell-1} \geq k)$, and hence,

$$\mathbb{E}[R_{k,n}] \;=\; 1 + \mathbb{E}\left[\sum_{\ell=1}^{H} R_\ell\right] \;=\; 1 + \sum_{\ell=1}^{H} 2^\ell\, \mathbb{P}(Z_{\ell-1} \geq k). \tag{5.1}$$

Similarly, a node $\mathbf{v}$ at level $\ell \in \{0, 1, \cdots, H\}$ receives a packet from the source and transmits it with probability p^ℓ. This gives the total expected number of transmissions for a transmission probability p to be

$$\mathbb{E}[T_{k,n}] \;=\; \sum_{\ell=0}^{H} \mathbb{E}[T_\ell] \;=\; n\frac{(2p)^{H+1}-1}{2p-1}.$$

Thus, $\mathbb{E}[T_{k,n}]$ is a monotonically increasing function of p, from which it can be inferred that

$$\tau_{k,n,\delta} = n\frac{(2p_{k,n,\delta})^{H+1}-1}{2p_{k,n,\delta}-1}. \tag{5.2}$$

Moreover, from (5.1) and the fact that $N = 2^{H+1}-1$, we have

$$p_{k,n,\delta} \;=\; \inf\left\{ p \;\middle|\; \frac{1+\sum_{\ell=1}^{H} 2^\ell\, \mathbb{P}(Z_{\ell-1}\geq k)}{2^{H+1}-1} \geq 1-\delta \right\},$$

where $Z_\ell \sim \mathrm{Bin}(n, p^\ell)$ for $\ell = 0, 1, \ldots, H-1$. The inequality within the expression for $p_{k,n,\delta}$ above can be rewritten as

$$\frac{\sum_{\ell=0}^{H-1} 2^{\ell+1}\mathbb{P}(Z_\ell \leq k-1)}{2^{H+1}-1} \leq \delta. \tag{5.3}$$

An analysis starting from (5.3) yields the two propositions below, which provide good lower and upper bounds on $p_{k,n,\delta}$. These bounds are plotted, for $k = 100$, $\delta = 0.1$ and $H = 50$, in Fig. 5.3(a) along with the exact values of $p_{k,n,\delta}$ obtained numerically from (5.3). The corresponding plots for $\tau_{k,n,\delta}$, obtained via (5.2), are shown in Fig. 5.3(b).

Proposition 5.2.1. *Let $k \geq 2$, $H \geq 2$, and $0 \leq \delta < \frac{1}{8}$ be fixed. For all $n \geq k$, we have $p_{k,n,\delta} > \left(\frac{k-1}{n}\right)^{\frac{1}{H-1}}$.*

In the case of $k = 1$ and $n > 1$, the lower bound can be improved to $p_{k,n,\delta} > \left(\frac{1}{n}\right)^{\frac{1}{H-1}}$.

Proposition 5.2.2. *Let $k \geq 2$, $H \geq 2$, and $0 < \delta \leq 1$ be fixed, and let $\delta' := \min\left\{\delta\left(\frac{2^{H+1}-1}{2^{H+1}-2}\right), 1\right\}$. Then, for all $n \geq 1$, we have*

$$p_{k,n,\delta} \leq \min\left\{\left(\frac{k-1+t}{n}\right)^{\frac{1}{H-1}}, 1\right\},$$

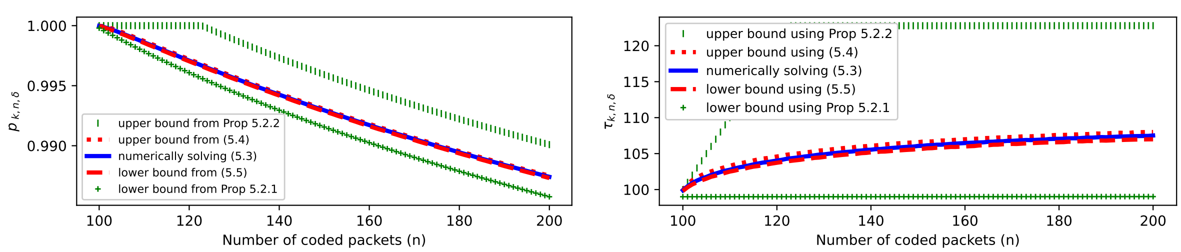
where $t = \sqrt{2(k-1)(-\ln\delta') + (\ln\delta')^2} - \ln\delta'$*. In the case of* $k = 1$*, the bound*

$$p_{k,n,\delta} \leq \min\left\{\left(\frac{-\ln\delta'}{n}\right)^{\frac{1}{H-1}}, 1\right\}$$

holds for all $n \geq 1$*.*

The following theorem, which summarizes the behaviour of $p_{k,n,\delta}$ on binary trees, is a direct consequence of Propositions 5.2.1 and 5.2.2.

Theorem 5.2.3. *Let* $k \geq 2$*,* $H \geq 2$ *and* $0 < \delta < \frac{1}{8}$ *be fixed. We then have* $p_{k,n,\delta} = \Theta\left(\left(\frac{k}{n}\right)^{\frac{1}{H-1}}\right)$*, where the constants implicit in the* Θ*-notation*[1] *may be chosen to depend only on* H *and* δ*.*



(a) Minimum retransmission probability

(b) Expected total number of transmissions

Figure 5.3: The middle curves are plots of the true values of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$ obtained from (5.3) and (5.2), for $k = 100$, $\delta = 0.1$ and $H = 50$. The other curves are bounds obtained via Propositions 5.2.1 and 5.2.2, (5.5), (5.4) and (5.2).

Tighter bounds for $p_{k,n,\delta}$ can be obtained by bounding the binomial cumulative distributive function (CDF) in (5.3) using Theorem A.2.1. This gives,

$$p_{k,n,\delta} \leq \inf\left\{p \;\middle|\; \frac{\sum_{\ell=0}^{H-1} 2^{\ell+1} C_{n,p^\ell}(k)}{2^{H+1}-1} \leq \delta\right\} \tag{5.4}$$

[1] *The notation* $a(n) = \Theta(b(n))$ *means that there are positive constants* c_1 *and* c_2 *such that* $c_1 b(n) \leq a(n) \leq c_2 b(n)$ *for all sufficiently large* n*.*

and

$$p_{k,n,\delta} \geq \inf \left\{ p \;\middle|\; \frac{\sum_{\ell=0}^{H-1} 2^{\ell+1} C_{n,p^\ell}(k-1)}{2^{H+1}-1} \leq \delta \right\}, \tag{5.5}$$

where $C_{n,q}(k) = \Phi\left(\operatorname{sgn}\left(\frac{k}{n} - q\right)\sqrt{2nD(\frac{k}{n} \,||\, q)}\right)$. The plots in Fig. 5.3 provide a theoretical explanation for why $\tau_{k,n,\delta}$ increases with n.

5.3 Proofs

In this section, we collect the proofs of Propositions 5.2.1 and 5.2.2.

Proposition 5.2.1. *Let $k \geq 2$, $H \geq 2$, and $0 \leq \delta < \frac{1}{8}$ be fixed. For all $n \geq k$, we have $p_{k,n,\delta} > \left(\frac{k-1}{n}\right)^{\frac{1}{H-1}}$.*

In the case of $k = 1$ and $n > 1$, the lower bound can be improved to $p_{k,n,\delta} > \left(\frac{1}{n}\right)^{\frac{1}{H-1}}$.

Proof of Prop. 5.2.1

Suppose that p is such that $np^{H-1} \leq k-1$. Then, Z_{H-1} has mean at most $k-1$. As a result, the median of Z_{H-1} is also at most $k-1$ [78, Corollary 3.1]. In other words, $\mathbb{P}(Z_{H-1} \leq k-1) \geq \frac{1}{2}$. Consequently,

$$\begin{aligned} \sum_{\ell=0}^{H-1} 2^{\ell+1}\, \mathbb{P}(Z_\ell \leq k-1) \;&\geq\; 2^H\, \mathbb{P}(Z_{H-1} \leq k-1) \\ &\geq\; 2^{H-1}, \end{aligned}$$

so that the left-hand side (LHS) of (5.3) is at least $\frac{2^{H-1}}{2^{H+1}-1} \geq \frac{2^{H-1}}{2^{H+1}} = 0.25 > \delta$. Hence, for (5.3) to hold, we must have $np^{H-1} > k-1$, from which the lower bound on $p_{k,n,\delta}$ follows.

In the case of $k = 1$, suppose that $p \leq \left(\frac{1}{n}\right)^{H-1}$. Then,

$$\begin{aligned}\mathbb{P}(Z_{H-1} = 0) &= \left(1 - p^{H-1}\right)^n \\ &\geq \left(1 - \frac{1}{n}\right)^n \\ &\geq \left(1 - \frac{1}{2}\right)^2 \\ &= 0.25,\end{aligned}$$

for all $n \geq 2$. Hence,

$$\sum_{\ell=0}^{H-1} 2^{\ell+1}\mathbb{P}(Z_\ell \leq k-1) \geq 2^H \mathbb{P}(Z_{H-1} = 0) \geq 2^{H-2}.$$

As a result, the LHS of (5.3) is at least $\frac{2^{H-2}}{2^{H+1}} = 0.125 > \delta$. Thus, again, for (5.3) to hold, we need $p > \left(\frac{1}{n}\right)^{H-1}$.

Proposition 5.2.2. *Let $k \geq 2$, $H \geq 2$, and $0 < \delta \leq 1$ be fixed, and let $\delta' := \min\left\{\delta\left(\frac{2^{H+1}-1}{2^{H+1}-2}\right), 1\right\}$. Then, for all $n \geq 1$, we have*

$$p_{k,n,\delta} \leq \min\left\{\left(\frac{k-1+t}{n}\right)^{\frac{1}{H-1}}, 1\right\},$$

where $t = \sqrt{2(k-1)(-\ln\delta') + (\ln\delta')^2} - \ln\delta'$. In the case of $k = 1$, the bound

$$p_{k,n,\delta} \leq \min\left\{\left(\frac{-\ln\delta'}{n}\right)^{\frac{1}{H-1}}, 1\right\}$$

holds for all $n \geq 1$.

Proof of Prop. 5.2.2

Note first that for all $\ell \leq H-1$, we have[2] $\mathbb{P}(Z_\ell \leq k-1) \leq \mathbb{P}(Z_{H-1} \leq k-1)$. Hence,

$$\begin{aligned}\sum_{\ell=0}^{H-1} 2^{\ell+1}\mathbb{P}(Z_\ell \leq k-1) &\leq \left(\sum_{\ell=0}^{H-1} 2^{\ell+1}\right)\mathbb{P}(Z_{H-1} \leq k-1) \\ &= (2^{H+1}-2)\mathbb{P}(Z_{H-1} \leq k-1).\end{aligned}$$

Thus, to show that (5.3) holds, it suffices to prove that $\mathbb{P}(Z_{H-1} \leq k-1) \leq \delta\left(\frac{2^{H+1}-1}{2^{H+1}-2}\right)$. It is, therefore, enough to show that $\mathbb{P}(Z_{H-1} \leq k-1) \leq \delta'$.

Consider $k=1$ first. Take $p = \min\left\{1, \left(\frac{C'}{n}\right)^{\frac{1}{H-1}}\right\}$, where $C' = -\ln\delta'$. Then,

$$\mathbb{P}(Z_{H-1} \leq k-1) = \mathbb{P}(Z_{H-1} = 0) = \left(1 - p^{H-1}\right)^n,$$

which, by choice of p, is either equal to 0 (if $C' \geq n$) or $(1 - C'/n)^n$ (if $C' < n$). In either case, $\mathbb{P}(Z_{H-1} = 0)$ is less than $e^{-C'} = \delta'$, as needed.

Consider $k \geq 2$ now. Take $p = \min\left\{1, \left(\frac{k-1+t}{n}\right)^{\frac{1}{H-1}}\right\}$, where t is as in the statement of the proposition. For $n \geq k-1+t$, we have $Z_{H-1} \sim \mathrm{Bin}(n, \frac{k-1+t}{n})$, so that

$$\begin{aligned}\mathbb{P}(Z_{H-1} \leq k-1) &= \mathbb{P}\big(Z_{H-1} \leq n(\tfrac{k-1+t}{n} - \tfrac{t}{n})\big) \\ &\leq e^{-n\,D(\frac{k-1}{n} \| \frac{k-1+t}{n})}\end{aligned}$$

via the Chernoff bound. Here, $D(\cdot \,\|\, \cdot)$ denotes the Kullback-Leibler divergence, defined as $D(x \,\|\, y) = x\ln\frac{x}{y} + (1-x)\ln\frac{1-x}{1-y}$. Using the bound $D(x \,\|\, y) \geq \frac{(x-y)^2}{2y}$, valid for $x \leq y$ [79], we further have

$$\mathbb{P}(Z_{H-1} \leq k-1) \leq e^{-n\left[\frac{(t/n)^2}{2(k-1+t)/n}\right]} = e^{-\frac{t^2}{2(k-1+t)}}.$$

Thus, to conclude that $\mathbb{P}(Z_{H-1} \leq k-1) \leq \delta'$, as required, it suffices to show that $\frac{t^2}{2(k-1+t)} \geq -\ln\delta'$. This can be re-written as $t^2 + 2t\ln\delta' + 2(k-1)\ln\delta' \geq 0$, or equivalently,

[2] This is easily shown by a standard coupling argument — see Lemma A.1.1 of Appendix A

$(t + \ln \delta')^2 + 2(k-1)\ln \delta' - (\ln \delta')^2 \geq 0$, which is evidently satisfied by our choice of t.

5.4 Conclusion

Our analysis extends easily to d-ary trees as well. Spherically symmetric trees in which every node at a particular level has the same number of children can also be handled with a similar analysis.

More generally, for any rooted tree, the condition for a near-broadcast equivalent to (5.3) is

$$\sum_{\ell=0}^{H-1} f_{\ell+1}\, \mathbb{P}(Z_\ell \leq k-1) \leq \delta,$$

where f_ℓ is the fraction of nodes of the tree at level ℓ. Notice that in the proofs of Proposition 5.2.1 and Proposition 5.2.2, we bound the required probabilities $\mathbb{P}(Z_\ell \leq k-1)$ with the probabilities $\mathbb{P}(Z_{H-1} \leq k-1)$. In particular, in the proof of Proposition 5.2.1, we replace the summation above with the term corresponding to $H-1$. Bounds obtained using this substitution are useful when the fraction of nodes at level H is the largest. In other words, the analysis given here carries over to trees in which the leaf nodes are present only at level H.

Nevertheless, our results indicate that introducing redundancy in the form of coding into the probabilistic retransmission protocol on tree-like structures is not beneficial in terms of the overall energy expenditure in the network. From our analysis, it is evident that the primary reason for this behaviour is the presence of a unique path from the root to any node on the tree. Introduction of additional coded packets is rendered useless due to the lack of multiple paths as in a grid. However, our treatment of the problem gives us an insight into the possible reason behind the difference in performance on trees and other well-connected graphs. This is the multipath diversity phenomenon described in Section 3.5. Characterizing it on different network topologies is not straightforward (see Section 9.1.2). Owing to this, in the following chapter, we investigate an alternate approach, using percolation theory, on well-connected graphs such as lattices and grids.

Chapter 6

Grids

In this chapter, we analyze the probabilistic forwarding mechanism with coded packets on grids. The main analytical tool used is the site percolation process. While we concentrate only on the square grid, we will see that the analysis carries over to more general lattice structures as well.

6.1 Square grid

Consider, for an odd integer $m > 1$, the discrete $m \times m$ grid $\Lambda_m := \left[-\frac{m-1}{2}, \frac{m-1}{2}\right]^2 \cap \mathbb{Z}^2$ centred at the origin. The source node is assumed to be at the centre of the grid. Simulation results for the probabilistic forwarding algorithm with $k = 100$ packets on the 31×31 grid (depicted in Fig. 6.1) are plotted in Fig. 6.2. In this section, we try to explain these observations by developing an analysis that is at least valid for large m. Specifically, we turn to the theory of site percolation on the integer lattice $\mathbb{Z}^2$ to explain the $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$ curves obtained via simulations on large grids Λ_m.

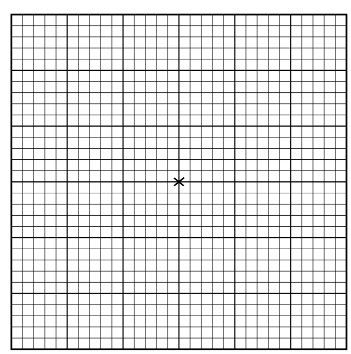
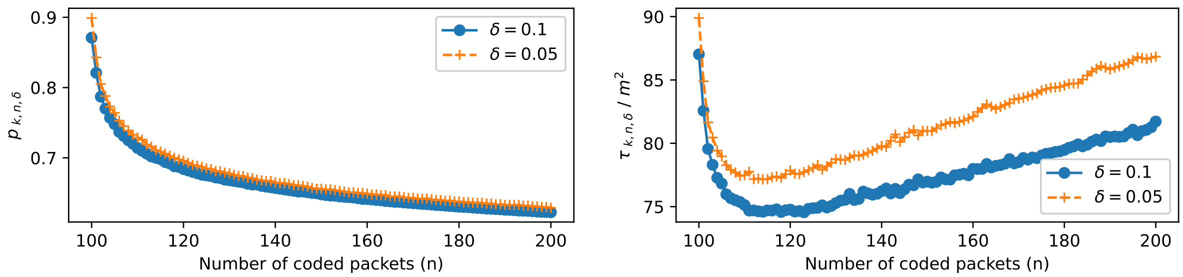


Figure 6.1: The source node ($\times$) is at the centre of the 31×31 grid.



(a) Minimum retransmission probability (b) Expected total number of transmissions

Figure 6.2: Simulations on a 31×31 grid. Probabilistic forwarding done with $k = 100$ packets.

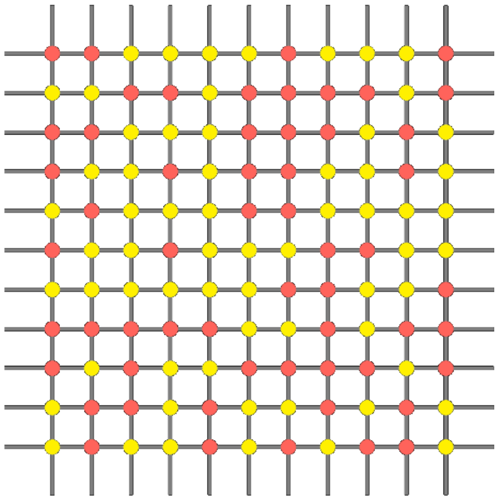
6.2 Preliminaries

6.2.1 Site percolation on $\mathbb{Z}^2$

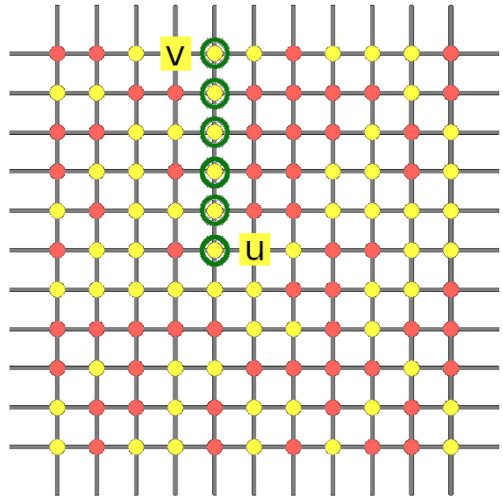
We start with a brief description of the site percolation process (see e.g. [80]) on $\mathbb{Z}^2$. Associate each vertex u of $\mathbb{Z}^2$ with a Bernoulli random variable, X_u, having parameter p. Then, site percolation is an i.i.d. process $(X_u)_{u \in \mathbb{Z}^2}$, with $X_u \sim \text{Ber}(p)$ for each $u \in \mathbb{Z}^2$, where the probability $p \in [0, 1]$ is a parameter of the process. A node or *site* $u \in \mathbb{Z}^2$ is *open* if $X_u = 1$, and is *closed* otherwise. An illustration is provided in Fig. 6.3(a). The product measure $\otimes_u \nu_u$, with $\nu_u \sim \text{Ber}(p)\ \forall\, u \in \mathbb{Z}^2$ is the push-forward measure of the $(X_u)_{u \in \mathbb{Z}^2}$ process on $\{0, 1\}^{\mathbb{Z}^2}$. We denote this by $\mathbb{P}_1$.

For $u = (u_x, u_y) \in \mathbb{Z}^2$, define $|u| := |u_x| + |u_y|$. Two sites u and v are joined by an edge, denoted by u—v, iff $|u - v| = 1$. The next few definitions are made with respect to a given realization of the process $(X_u)_{u \in \mathbb{Z}^2}$. Two sites u and v are connected by an *open path*, denoted by $u \longleftrightarrow v$, if there is a sequence of sites $u_0 = u, u_1, u_2, \ldots, u_n = v$ such that u_k is open for all $k \in \{0, 1, \ldots, n\}$ and u_{k-1}—u_k for all $k \in [n]$ (see Fig. 6.3(b)).

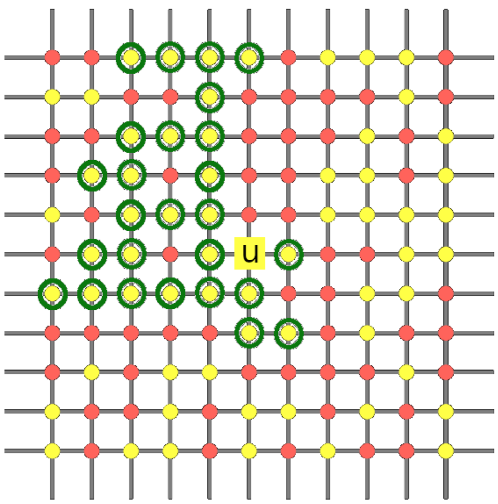
The *open cluster*, C_u, containing the site u is defined as $C_u = \{v \in \mathbb{Z}^2 | u \longleftrightarrow v\}$. Thus, C_u consists of all sites connected to u by open paths, as shown in Fig 6.3(c). In particular, $C_u = \emptyset$ if u is itself closed. The *boundary*, ∂C_u, of a non-empty open cluster C_u is the set of all closed sites $v \in \mathbb{Z}^2$ such that v—w for some $w \in C_u$. The set $C_u^{\text{ext}} := C_u \cup \partial C_u$ is called



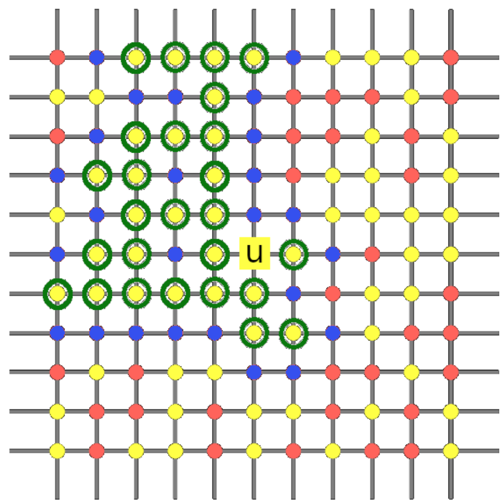
(a) ● - open, ● - closed



(b) $u \longleftrightarrow v$



(c) Encircled nodes form C_u



(d) Encircled nodes with ● nodes form C_u^{ext}

Figure 6.3: Site percolation on $\mathbb{Z}^2$.

an *extended cluster* (see Fig 6.3(d)). The cluster C_u (resp. C_u^{ext}) is termed an *infinite open cluster (IOC)* (resp. *infinite extended cluster (IEC)*) if it has infinite cardinality. Note that C_u^{ext} is infinite iff C_u is infinite.

It is well-known that there exists a *critical probability* $p_c \in (0, 1)$ such that for all $p < p_c$, there is almost surely (with respect to $\mathbb{P}_1$) no IOC, while for all $p > p_c$, there is almost surely a unique IOC. We do not know what happens at $p = p_c$, as the exact value of p_c is itself not known (for site percolation on $\mathbb{Z}^2$). It is believed that $p_c \approx 0.59$ [80, Chapter 1]. Another quantity of interest, which will play a crucial role in our analysis, is the *percolation probability* $\theta(p)$, defined to be the probability that the origin $\mathbf{0}$ is in an IOC. In our analysis, we also consider the probability, $\theta^{\text{ext}}(p)$, of the origin $\mathbf{0}$ being in an IEC. Clearly, from our definition of the IEC, for $p < p_c$, we have $\theta^{\text{ext}}(p) = \theta(p) = 0$; for $p > p_c$, it is not difficult to see that $\theta^{\text{ext}}(p) \geq \theta(p) > 0$. It is known that $\theta(p)$ is non-decreasing and infinitely differentiable in the region $p > p_c$ [81], but there is no analytical expression known for it. The following lemma, outlined in [60], expresses $\theta^{\text{ext}}(p)$ in terms of $\theta(p)$.

Lemma 6.2.1. *For any $p > p_c$, we have $\theta^{ext}(p) = \dfrac{\theta(p)}{p}$.*

Proof. Let C and C^{ext} be the (unique) IOC and IEC, respectively. We then have

$$\theta(p) = \mathbb{P}_1(\mathbf{0} \in C) = \mathbb{P}_1(\mathbf{0} \in C^{\text{ext}} \text{ and } \mathbf{0} \text{ is open}). \tag{6.1}$$

Now, observe that the event $\{\mathbf{0} \in C^{\text{ext}}\}$ is determined purely by the states of the nodes other than the origin. If at least one neighbour of the origin is present in C, then $\mathbf{0} \in C^{\text{ext}}$. Hence, this event is independent of the event that $\mathbf{0}$ is open. Thus, the RHS of (7.27) equals $\theta^{\text{ext}}(p) \cdot p$, which proves the lemma. $\square$

Note that, since the origin can be part of the IOC only if it is open, we have that $\theta(p) \leq p$ which ensures that $\theta^{\text{ext}}(p) \leq 1$. Fig. 6.4 plots $\theta(p)$ and $\theta^{\text{ext}}(p)$ as functions of p, the former being obtained via simulations based on Theorem 6.2.3.

6.2.2 Ergodic theorems

Let A be a finite alphabet, and ν a probability measure on it. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathsf{A}^{\mathbb{Z}^2}$, $\mathcal{F}$ is the σ-algebra of cylinder sets, and $\mathbb{P}$ is the product measure $\otimes_u \nu_u$ with $\nu_u = \nu$ for all $u \in Z^2$. For $z \in \mathbb{Z}^2$, define the shift operator $T_z : \Omega \to \Omega$ that maps $\omega = (\omega_u)_{u \in \mathbb{Z}^2}$ to $T_z\omega$ such that $(T_z\omega)_u = \omega_{u-z}$ for all $u \in \mathbb{Z}^2$. Correspondingly, for a random variable X defined on this probability space, set $T_z X := X \circ T_{-z}$, i.e., $(T_z X)(\omega) = X(T_{-z}\omega)$ for all $\omega \in \Omega$.

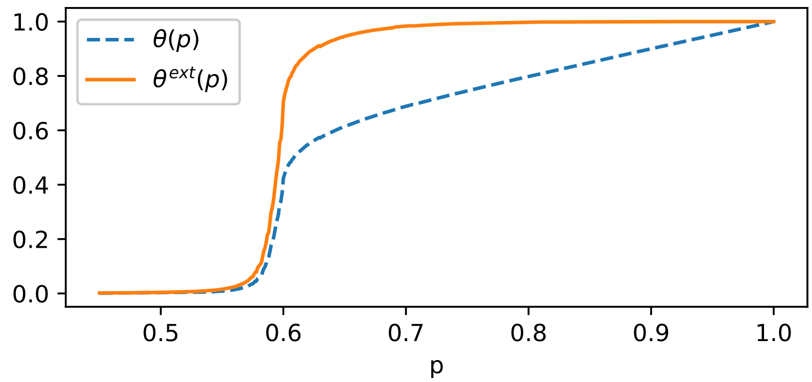


Figure 6.4: $\theta(p)$ and $\theta^{\text{ext}}(p)$ vs. p

The following theorem is a special case of Tempelman's pointwise ergodic theorem (see e.g., [82, Chapter 6]). For $\mathsf{A} = \{0, 1\}$, this was stated as Proposition 8 in [83].

Theorem 6.2.2. *For any random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite mean, we have*

$$\lim_{m\to\infty} \frac{1}{m^2} \sum_{z\in\Lambda_m} T_z X = \mathbb{E}[X] \qquad \mathbb{P}\text{-}a.s.,$$

where $\Lambda_m := [-\frac{m-1}{2}, \frac{m-1}{2}]^2 \cap \mathbb{Z}^2$ is the $m \times m$ grid (m odd).

In a broad sense, ergodic theorems relate a quantity averaged over space (LHS) to the time-average (expectation on the RHS). The expectations are often easy to compute and thus, these theorems can be used to obtain approximations for spatial averages in the limit of large m.

Site percolation

The theorem applies to the case of site percolation, in which ν above is the Bernoulli(p) measure on $\mathsf{A} = \{0, 1\}$. Applying the theorem with $X = \mathbb{1}_{\{\mathbf{0}\in C\}}$, the indicator function of $\mathbf{0}$ being in the (unique when $p > p_c$) IOC C, and again with $X = \mathbb{1}_{\{\mathbf{0}\in C^{\text{ext}}\}}$, we obtain the following theorem.

Theorem 6.2.3. *Let $p > p_c$, and let C and C^{ext}, respectively, be the (almost surely) unique IOC and IEC of a site percolation process on $\mathbb{Z}^2$ with parameter p. Then, almost* *surely, we have*

$$\lim_{m\to\infty} \frac{1}{m^2}|C \cap \Lambda_m| = \theta(p) \qquad \textit{and} \qquad \lim_{m\to\infty} \frac{1}{m^2}|C^{ext} \cap \Lambda_m| = \theta^{ext}(p).$$

Using the dominated convergence theorem (DCT), we also have

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}|C \cap \Lambda_m|\right] = \theta(p) \qquad \text{and} \qquad \lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}|C^{\text{ext}} \cap \Lambda_m|\right] = \theta^{\text{ext}}(p).$$

Based on the first equation above, to obtain an estimate of $\theta(p)$, the site percolation process with parameter p was simulated on a 1001×1001 grid and the average fraction of nodes (averaged over 100 realizations of the process) in the largest open cluster was taken to be the value of $\theta(p)$. These are the values of $\theta(p)$ plotted in Fig. 6.4. We would like to emphasize that the plots in the figure should only be trusted for $p > p_c$, as Theorem 6.2.3 is only valid in that range. However, as the exact value of p_c is unknown, simulation results are reported for the range of p values shown in the plot. We will use values from this plot in our numerical results.

Multiple site percolations on $\mathbb{Z}^2$

Now, consider n independent site percolation processes on $\mathbb{Z}^2$, with parameter $p > p_c$. Let O denote the event that the origin is open in all n percolations. We will use $\mathbb{P}^{\text{o}}$ and $\mathbb{E}^{\text{o}}$, respectively, to denote the probability measure and expectation operator conditioned on the event O, and $\mathbb{P}$ and $\mathbb{E}$ for the unconditional versions of these. Since $p > p_c$, each percolation has a unique IOC and IEC, almost surely with respect to $\mathbb{P}$ ($\mathbb{P}$-a.s.). Next, with $\mathsf{A} = \{0,1\}^n$ and ν the product of n independent Bernoulli(p) measures, we are in the setting of n independent site percolations on $\mathbb{Z}^2$. Let $C_{k,n}^{\text{ext}}$ be the set of sites that belong to the IEC in at least k out of the n percolations. In this case, taking E to be the event that the $\mathbf{0}$ is in the IEC in at least k of the n independent percolations and $X = \mathbb{1}_E$, and applying Theorem 6.2.2, we obtain

$$\lim_{m\to\infty} \frac{1}{m^2}|C_{k,n}^{\text{ext}} \cap \Lambda_m| = \mathbb{P}(E) \qquad \mathbb{P}\text{-a.s.}$$

Using the fact that the origin is in the IEC with probability $\theta^{\text{ext}}(p)$, and since all the n percolations are independent, the probability of the event E on the RHS in the above equation is the tail probability of a $Bin(n, \theta^{\text{ext}}(p))$ distributed random variable. More precisely

$$\mathbb{P}(E) = \sum_{j=k}^{n} \binom{n}{j} (\theta^{\text{ext}}(p))^j (1 - \theta^{\text{ext}}(p))^{n-j}.$$

In the following, we will denote this by $\theta^{\text{ext}}_{k,n}(p)$. Thus, we have

Theorem 6.2.4. *For $p > p_c$,*

$$\lim_{m \to \infty} \frac{1}{m^2} \, |C^{ext}_{k,n} \cap \Lambda_m| = \theta^{ext}_{k,n}(p) \qquad \mathbb{P}\text{-}a.s.$$

6.3 Analysis and main results

Having developed the required background, in this section, we now map the probabilistic forwarding mechanism on the finite grid to the probabilistic forwarding mechanism on the infinite $\mathbb{Z}^2$ lattice. We then obtain estimates for the minimum forwarding probability and the expected total number of transmissions by relating it to the site percolation process.

6.3.1 Relating site percolation to probabilistic forwarding

Site percolation on $\mathbb{Z}^2$ is a faithful model for probabilistic forwarding of a single packet on the infinite lattice $\mathbb{Z}^2$. The origin $\mathbf{0}$ is the source of the packet. The open cluster, $C_{\mathbf{0}}$, containing the origin $\mathbf{0}$ corresponds to the set of nodes that transmit (forward) the packet, and the extended cluster $C^{\text{ext}}_{\mathbf{0}}$ corresponds to the set of nodes that receive the packet. The only caveat is that, since the source is assumed to always transmit the packet, we must consider only those realizations of the site percolation process in which the origin $\mathbf{0}$ is open. In other words, we must consider the site percolation process, conditioned on the event that the origin is open. By extension, the probabilistic forwarding of n coded packets corresponds to n independent site percolation processes on $\mathbb{Z}^2$, conditioned on the event that the origin is open in all n percolations.

Let O denote the event that the origin is open in all n percolations. In our analysis, we will use $\mathbb{P}^o$ and $\mathbb{E}^o$, respectively, to denote the probability measure and expectation operator conditioned on the event O, and $\mathbb{P}$ and $\mathbb{E}$ for the unconditional versions of these.

6.3.2 Analysis of probabilistic forwarding on a large (finite) grid

In this section, we analyze the probabilistic forwarding mechanism on the finite grid Λ_m using the following approach. We map the probabilistic forwarding mechanism on Λ_m onto the probabilistic forwarding mechanism on the infinite $\mathbb{Z}^2$ lattice. From the discussion in the previous subsection, this is nothing but n independent site percolations on $\mathbb{Z}^2$ conditioned on the event O. Using ergodic theorems for the site percolation process, we get a handle on the expected number of nodes that receive at least k out of the n packets from the origin on $\mathbb{Z}^2$. This, in turn, is used to obtain estimates of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$. In our upcoming analysis, we make the following assumption.

Assumption 1. *We operate in the super-critical region for site percolation on $\mathbb{Z}^2$, i.e. $p > p_c$.*

We provide a justification for this assumption in Section 6.6.1.

Minimum forwarding probability

Denote by $R_{k,n}(\Lambda_m)$, the number of successful receivers in Λ_m, i.e., the number of nodes that receive at least k out of n packets during the probabilistic forwarding mechanism on Λ_m. The following theorem is our main result for grids. Its proof is quite technical, and is presented in the next section.

Theorem 6.3.1. *For $p > p_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] = \sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^{ext}(p))^{t+j}(1-\theta^{ext}(p))^{n-j}.$$

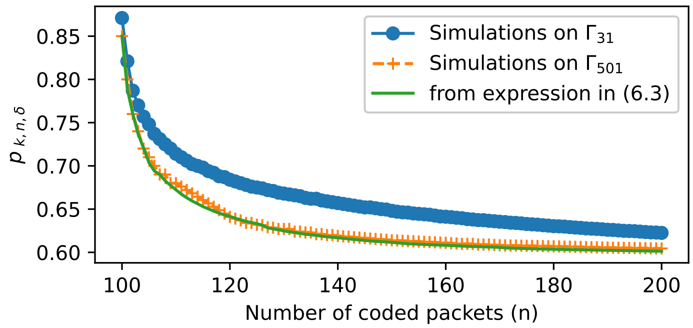


Figure 6.5: Comparison of the minimum forwarding probability obtained via simulations on a 31×31 grid and a 501×501 grid, with the results obtained numerically from (6.3), for $k = 100$ data packets and $\delta = 0.1$.

Equivalently,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] = \mathbb{P}(Y \geq k), \tag{6.2}$$

where $Y \sim \mathrm{Bin}(n, (\theta^{ext}(p))^2)$.

Thus, for k, n, δ fixed, we have for all sufficiently large grids Λ_m,

$$p_{k,n,\delta}(\Lambda_m) \approx \inf\{p \mid Pr(Y \geq k) \geq 1-\delta\}, \tag{6.3}$$

where $Y \sim \mathrm{Bin}(n, (\theta^{\mathrm{ext}}(p))^2)$. This can be evaluated numerically using the values of $\theta^{\mathrm{ext}}(p)$ plotted in Fig. 6.4. For large k and n, the probability $\mathbb{P}(Y \geq k)$ can be approximated well using the bounds given in Theorem A.2.1 in the appendix. A sample of results thus obtained are shown in Fig. 6.5. It is clear that these results match very well with those obtained from simulations on a 501×501 grid.

Total number of transmissions

We next look into estimating the expected total number of transmissions at a given forwarding probability p. Consider the transmission of a single packet on the finite grid Λ_m. Let $T(\Lambda_m)$ be the number of transmissions of the packet on the finite grid Λ_m and let $\mathcal{T}(\mathbb{Z}^2) \cap \Lambda_m$ be the set of nodes in Λ_m which receive the packet from the origin and

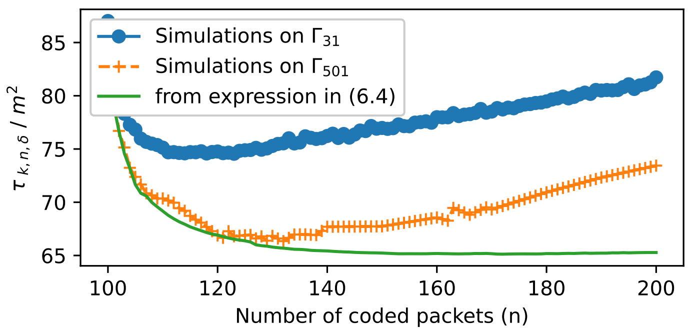


Figure 6.6: Comparison of the expected total number of transmissions, normalized by the grid size m^2, obtained via simulations on Γ_{31} and Γ_{501}, with the expression from (6.4), for $k = 100$ data packets and $\delta = 0.1$.

transmit it on the infinite $\mathbb{Z}^2$ lattice. It can be shown[1] that

$$\lim_{m\to\infty} \frac{\mathbb{E}[T(\Lambda_m)]}{m^2} = \lim_{m\to\infty} \frac{\mathbb{E}\left[|\mathcal{T}(\mathbb{Z}^2) \cap \Lambda_m|\right]}{m^2}.$$

Now, $\mathcal{T}(\mathbb{Z}^2)$ is simply the open cluster $C_{\mathbf{0}}$ in the percolation framework. Thus, when normalized by the grid size m^2, the expected number of transmissions, $\mathbb{E}[T(\Lambda_m)]$, for probabilistic forwarding on a large (but finite) grid Λ_m is well-approximated by $\mathbb{E}\big[|C_{\mathbf{0}} \cap \Lambda_m| \;\big|\; \mathbf{0} \text{ is open}\big]$. The following lemma gives an expression for this quantity in the limit as the grid size goes to infinity.

Lemma 6.3.2. *For site percolation with $p > p_c$, we have*

$$\lim_{m\to\infty} \frac{1}{m^2}\mathbb{E}\big[|C_{\mathbf{0}} \cap \Lambda_m| \;\big|\; \mathbf{0} \textit{ is open}\big] \;=\; \frac{\theta(p)^2}{p}.$$

Proof. We use $\mathbb{P}^{\mathbf{0}}$ and $\mathbb{E}^{\mathbf{0}}$, respectively, to denote the probability measure and expectation operator conditioned on the event that the origin $\mathbf{0}$ is open. Let C be the (unique) IOC,

[1] This is shown using arguments entirely analogous to those used to show (6.6) in Section 6.4. We omit the details.

and A the event $\{\mathbf{0} \in C\}$. Then,

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{1}{m^2}|C_{\mathbf{0}} \cap \Lambda_m|\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}|C_{\mathbf{0}} \cap \Lambda_m| \;\middle|\; A\right] \mathbb{P}^{\mathbf{0}}(A) + \lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{1}{m^2}|C_{\mathbf{0}} \cap \Lambda_m| \;\middle|\; A^c\right] \mathbb{P}^{\mathbf{0}}(A^c)$$

Now, given A^c (i.e., $\mathbf{0} \notin C$), $C_{\mathbf{0}}$ is $\mathbb{P}^{\mathbf{0}}$-a.s. finite, and so by the DCT,

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{1}{m^2}|C_{\mathbf{0}} \cap \Lambda_m| \;\middle|\; A^c\right] = 0.$$

On the other hand, given A, we have $C_{\mathbf{0}} = C$. From Theorem 6.2.3, we know that $\lim\limits_{m\to\infty} \frac{1}{m^2}|C \cap \Lambda_m| = \theta(p)$ $\mathbb{P}_1$-a.s.. Moreover, this statement holds even when the probability measure $\mathbb{P}_1$ is conditioned on A, since $\mathbb{P}_1(A) = \theta(p) > 0$ for $p > p_c$. So, again by the DCT, $\lim\limits_{m\to\infty} \mathbb{E}[\frac{1}{m^2}|C \cap \Lambda_m| \mid A] = \theta(p)$. We have thus shown that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{1}{m^2}|C_{\mathbf{0}} \cap \Lambda_m|\right] = \theta(p)\, \mathbb{P}^{\mathbf{0}}(A).$$

The proof is completed by observing that $\mathbb{P}^{\mathbf{0}}(A) = \frac{\mathbb{P}_1(A)}{\mathbb{P}_1(\mathbf{0} \text{ is open})} = \frac{\theta(p)}{p}$. □

Thus, in probabilistic forwarding of a single packet on a large grid Λ_m, the expected number of transmissions, normalized by the grid size m^2, is approximately $\frac{\theta(p)^2}{p}$. Hence, when we have n coded packets, by linearity of expectation, the expected total number of transmissions, again normalized by the grid size m^2, is approximately $n\,\frac{\theta(p)^2}{p}$. In particular, setting $p = p_{k,n,\delta}$, we obtain

$$\frac{1}{m^2}\,\tau_{k,n,\delta}(\Lambda_m) \;\approx\; n\frac{\theta(p_{k,n,\delta})^2}{p_{k,n,\delta}}, \tag{6.4}$$

provided that $p_{k,n,\delta} > p_c$.

Fig. 6.6 compares, for $k = 100$ data packets and $\delta = 0.1$, the values of $\frac{1}{m^2}\tau_{k,n,\delta}$ obtained using (6.4), (6.3) and the $\theta(p)$ values from Fig. 6.4, with those obtained via simulations on the Γ_{31} and Γ_{501} grids. The curve based on (6.4), (6.3) and $\theta(p)$ initially tracks the

Γ_{501} curve well, but trails off after $n = 130$. This is because the former curve uses the approximation for $p_{k,n,\delta}$ in (6.3), which, for any given n, is valid only for sufficiently large m. For values of n larger than 130, $m = 501$ may not fall in the "sufficiently large" range. This is discussed in more detail in Section 6.6.2.

Moreover, it can be verified that the mismatch between the curve for the simulations on Λ_{501} and the curve obtained using (6.4) in Fig. 6.6 is solely due to the difference between the corresponding curves in Fig. 6.5. If instead of using the $p_{k,n,\delta}$ values obtained from (6.3), we were to use those from the simulations on Λ_{501}, then the curves match as seen in Fig. 6.7

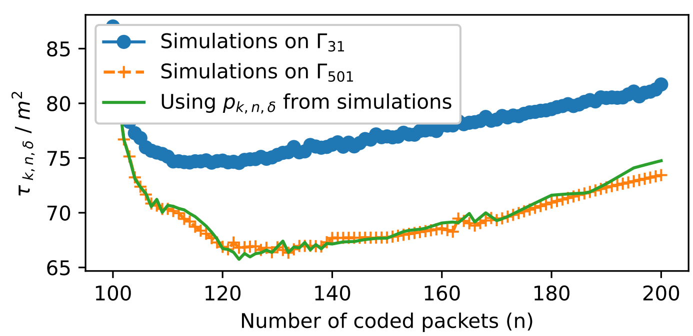
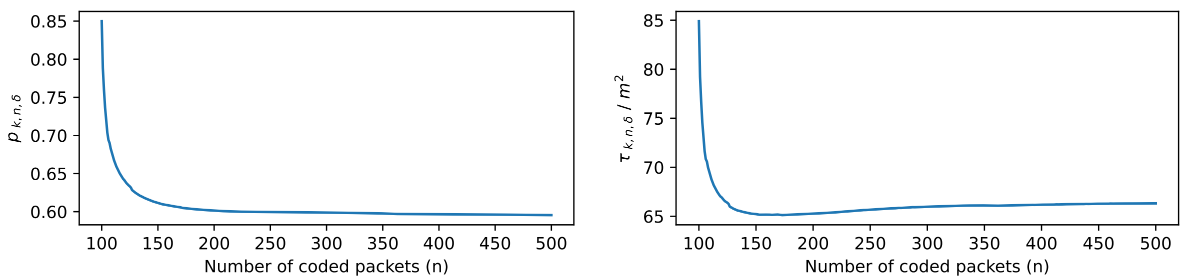


Figure 6.7: Comparison of the expected total number of transmissions, normalized by the grid size m^2, obtained via simulations on Γ_{31} and Γ_{501}, with the expression from (6.4) using $p_{k,n,\delta}$ values from simulations on Λ_{501}, for $k = 100$ data packets and $\delta = 0.1$.

Nonetheless, it is instructive to note that, for fixed values of k and δ, the expression on the RHS of (6.4) is indeed minimized for some n. This can be verified numerically by plotting the RHS of (6.4) using the values of $\theta(p)$ from Fig. 6.4 and the approximation to $p_{k,n,\delta}$ in (6.3) for a larger range of n values. Plots for $k = 100$, $\delta = 0.1$ and n varying from 100 to 500 are shown in Fig. 6.8. Observe that the curve plotted in Fig. 6.8(b) is decreasing in n till $n \approx 180$, and it increases thereafter, albeit very slowly. This indicates that, for $k = 100$ and $\delta = 0.1$, the expected number of transmissions $\tau_{k,n,\delta}(\Lambda_m)$ is minimized at $n \approx 180$ for all sufficiently large grids Λ_m. Thus, our analysis provides theoretical validation, at least for large grids, for the observed behaviour of $\tau_{k,n,\delta}$ as a function of n, and indicates a benefit to introducing some coding into the probabilistic forwarding mechanism on grids.



(a) Minimum retransmission probability

(b) Expected total number of transmissions normalized by the grid size m^2.

Figure 6.8: The minimum forwarding probability is numerically computed from (6.3) and the expected number of transmissions is obtained via (6.4), for $k = 100$ data packets and $\delta = 0.1$.

6.4 Proofs

In this section, we will provide the proof of Theorem 6.3.1. This will involve relating the probabilistic forwarding mechanism on the finite grid to the site percolation mechanism on the infinite grid. Asymptotic results for the fraction of successful receivers and the expected total number of transmissions are then obtained using ergodic theorems.

Let $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ denote the set of all nodes that receive at least k of the n coded packets during the probabilistic forwarding protocol on $\mathbb{Z}^2$. As a first step, we will show that $R_{k,n}(\Lambda_m)$ and $|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m|$ are the same in expectation, in the limit as the grid size, m, goes to infinity. In general, it is only true that $R_{k,n}(\Lambda_m)$ is stochastically dominated[2] by $|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m|$, since a node in $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$ could receive packets from the origin through paths in $\mathbb{Z}^2$ that do not lie entirely within Λ_m.

In the percolation jargon (on $\mathbb{Z}^2$), $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$ comprises those nodes of Λ_m that are in the extended cluster containing the origin ($C_{\mathbf{0}}^{\text{ext}}$) in at least k out of n percolations.

[2] A random variable X is stochastically dominated by a random variable Y if $\mathbb{P}(X \geq x) \leq \mathbb{P}(Y \geq x)$ for all $x \in \mathbb{R}$. For non-negative random variables, this implies that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Recall that a node u is in $C_{\mathbf{0}}^{\text{ext}}$ if either the node u or some one-hop neighbour of u is connected to the origin through an open path. Call such an open path a *conduit* (for a packet) from the origin to u. If a conduit lies completely within Λ_m, we call it a Λ_m*-conduit*. We also say that, if vertex u has a conduit, it is necessarily in $C_{\mathbf{0}}^{\text{ext}}$.

The nodes in $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$ may have received some packets from the origin through Λ_m-conduits, and some others through conduits that go outside Λ_m. We are interested in the former, since, when operating on a finite grid Λ_m, nodes of $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$ without Λ_m-conduits cannot be successful receivers in Λ_m. More precisely, we are interested in those nodes of Λ_m which are part of the extended cluster containing the origin through at least one Λ_m-conduit, in at least k out of the n percolations. Note that these are the nodes that receive at least k out of the n packets in the finite grid model; we denote this collection of nodes by $\mathcal{R}_{k,n}(\Lambda_m)$. Thus, $|\mathcal{R}_{k,n}(\Lambda_m)| = R_{k,n}(\Lambda_m)$. We denote the remaining nodes by $\overline{\mathcal{R}}_{k,n}(\Lambda_m) := (\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m) \backslash \mathcal{R}_{k,n}(\Lambda_m)$. Thus, $\mathcal{R}_{k,n}(\Lambda_m)$ and $\overline{\mathcal{R}}_{k,n}(\Lambda_m)$ form a partition of $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$, i.e.,

$$\mathcal{R}_{k,n}(\Lambda_m) \cap \overline{\mathcal{R}}_{k,n}(\Lambda_m) = \emptyset$$

and

$$\mathcal{R}_{k,n}(\Lambda_m) \cup \overline{\mathcal{R}}_{k,n}(\Lambda_m) = \mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m. \tag{6.5}$$

Note that any node in $\overline{\mathcal{R}}_{k,n}(\Lambda_m)$ has the property that for at least one of the packets it receives, any conduit through which it receives that packet *necessarily* goes outside Λ_m. Such a node is said to *receive at least one packet from outside* Λ_m. It does not receive this packet through any Λ_m-conduit.

We first show that the expected fraction of nodes in Λ_m that receive at least one packet from outside Λ_m vanishes asymptotically with the grid size m. In this direction, we will need the following definition: For $0 < \epsilon < 4$, let

$$\Lambda_{m,\epsilon} := \left\{ \begin{array}{cc} \Lambda_{\left\lfloor m\sqrt{1-\frac{\epsilon}{4}} \right\rfloor}, & \text{if } \left\lfloor m\sqrt{1-\frac{\epsilon}{4}} \right\rfloor \text{ is odd} \\ \Lambda_{\left\lfloor m\sqrt{1-\frac{\epsilon}{4}} \right\rfloor - 1}, & \text{if } \left\lfloor m\sqrt{1-\frac{\epsilon}{4}} \right\rfloor \text{ is even} \end{array} \right\}$$

Recall that Λ_m was defined as $\Lambda_m := \left[-\frac{m-1}{2}, \frac{m-1}{2}\right]^2 \cap \mathbb{Z}^2$ when m was odd. We will think of $\Lambda_{m,\epsilon}$ as being $\Lambda_{m\sqrt{1-\frac{\epsilon}{4}}}$ in our calculations, and hence the number of nodes in $\Lambda_{m,\epsilon}$ is approximately $m^2\left(1-\frac{\epsilon}{4}\right)$. We now have the following lemma.

Lemma 6.4.1. *For $p > p_c$, we have*

$$\lim_{m\to\infty} \frac{1}{m^2}\, \mathbb{E}^o\left[|\overline{\mathcal{R}}_{k,n}(\Lambda_m)|\right] = 0$$

Since $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$ is a disjoint union of nodes in $\mathcal{R}_{k,n}(\Lambda_m)$ and $\overline{\mathcal{R}}_{k,n}(\Lambda_m)$, the previous lemma shows that

$$\lim_{m\to\infty} \frac{1}{m^2}\mathbb{E}^o\left[|\mathcal{R}_{k,n}(\Lambda_m)|\right] = \lim_{m\to\infty} \frac{1}{m^2}\mathbb{E}^o\left[|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m|\right], \tag{6.6}$$

This provides us with a mapping between the probabilistic forwarding mechanism on a large (but finite) grid Λ_m and the infinite lattice $\mathbb{Z}^2$.

In our analysis on the grid, we will be interested in the expected value of $|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m)|$ when conditioned on the event A_T^{ext}, defined, for any $T \subset [n]$, as the event that the origin is in the IEC in exactly the percolations indexed by T. As a corollary of Lemma 6.4.1, we also obtain

Corollary 6.4.2. *For $p > p_c$, we have*

$$\lim_{m\to\infty} \frac{1}{m^2}\mathbb{E}^o\left[\overline{\mathcal{R}}_{k,n}(\Lambda_m)\middle| A_T^{ext}\right] = 0.$$

We defer the proofs of Lemma 6.4.1 and Corollary 6.4.2 to the next section. Indeed, it has to be shown that the conditioning done in Corollary 6.4.2 is valid, which is done in Proposition 6.5.1.

To analyze $|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m|$, we now use the ergodic theorem for n independent copies of the site percolation process on $\mathbb{Z}^2$. Recall that from Theorem 6.2.4, we had

$$\lim_{m\to\infty} \frac{1}{m^2}|C_{k,n}^{\text{ext}} \cap \Lambda_m| = \theta_{k,n}^{\text{ext}}(p) = \sum_{j=k}^{n} \binom{n}{j} (\theta^{\text{ext}}(p))^j (1-\theta^{\text{ext}}(p))^{n-j}.$$

From this, we derive a useful fact that plays a key role in our analysis. Since the event, say A_n, that the origin is in the IOC in all n percolations has positive probability ($\theta(p)^n > 0$ for $p > p_c$), the theorem statement also holds almost surely when conditioned on A_n. Hence, by the DCT, we also have

Corollary 6.4.3.

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}|C_{k,n}^{ext} \cap \Lambda_m| \;\middle|\; A_n\right] = \theta_{k,n}^{ext}(p)\,.$$

We are now in a position to prove Theorem 6.3.1, which is restated below for convenience. The proof is obtained by carefully relating $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ to the set $C_{k,n}^{\text{ext}}$, and then using Corollary 6.4.3.

Theorem 6.3.1. *For $p > p_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \;=\; \sum_{t=k}^{n}\sum_{j=k}^{t} \binom{n}{t}\binom{t}{j} (\theta^{ext}(p))^{t+j}(1-\theta^{ext}(p))^{n-j}.$$

Equivalently,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \;=\; \mathbb{P}(Y \ge k), \tag{6.2}$$

where $Y \sim \mathrm{Bin}(n, (\theta^{ext}(p))^2)$.

Proof. Before we begin, recall from (6.5) that $\mathcal{R}_{k,n}(\Lambda_m)$ and $\overline{\mathcal{R}}_{k,n}(\Lambda_m)$ form a partition of $\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m$. In the framework of n independent site percolations, $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ is the set of sites in $\mathbb{Z}^2$ that are in the extended cluster containing the origin in at least k of the n percolations (conditioned on the origin being open).

We start with

$$\mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,n}(\Lambda_m)|\right] = \sum_{t=0}^{n} \sum_{\substack{T\subseteq[n]:\\|T|=t}} \mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,n}(\Lambda_m)| \;\middle|\; A_T^{\mathrm{ext}}\right] \mathbb{P}^{\mathrm{o}}(A_T^{\mathrm{ext}}). \tag{6.7}$$

Our approach in the ensuing discussion would be to first obtain results for $\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m$, and then transfer them to $\mathcal{R}_{k,n}(\Lambda_m)$. Motivated by our discussion following Lemma 6.4.1, consider the summand of (6.7) with $\mathcal{R}_{k,n}(\Lambda_m)$ replaced by $\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m$, i.e., $\mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right]$.

Suppose that $|T| = t < k$. Given A_T^{ext}, the origin is in the IEC in no more than $k-1$ of the percolations; hence, each site in $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ must belong to the finite cluster, denoted by $C_{\mathbf{0}}[j]$, in the jth percolation, for some $j \notin T$. As a result, given A_T^{ext}, $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ is contained in the union $\cup_{j\notin T} C_{\mathbf{0}}[j]$, which is finite $\mathbb{P}^{\mathrm{o}}$-a.s, so that $\lim_{m\to\infty} \frac{1}{m^2}\,|\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m| = 0$ $\mathbb{P}^{\mathrm{o}}$-a.s.. Since $\mathcal{R}_{k,n}(\Lambda_m) \subseteq \mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m$, we also obtain $\lim_{m\to\infty} \frac{R_{k,n}(\Lambda_m)}{m^2} = 0$ $\mathbb{P}^{\mathrm{o}}$-a.s.. Consequently, by the DCT, we have for any $T\subseteq[n]$ with $|T| < k$,

$$\lim_{m\to\infty} \mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] = 0 \qquad \text{and} \qquad \lim_{m\to\infty} \mathbb{E}^{\mathrm{o}}\left[\frac{|\mathcal{R}_{k,n}(\Lambda_m)|}{m^2} \;\middle|\; A_T^{\mathrm{ext}}\right] = 0. \tag{6.8}$$

Next, consider any summand in (6.7) with $|T| = t \geq k$ and $\mathcal{R}_{k,n}(\Lambda_m)$ replaced by $\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m$ as before. The sites in $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ can be exactly one of two types: those that belong to the extended cluster $C_{\mathbf{0}}^{\mathrm{ext}}$ in at least k of the percolations indexed by T; and those that do not. Let $\mathcal{R}_{k,T}$ be the subset of $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ consisting of sites of the first type, and let $\mathcal{Q} = \mathcal{R}_{k,n}(\mathbb{Z}^2)\setminus\mathcal{R}_{k,T}$. Thus,

$$\mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] \;=\; \mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,T}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] + \mathbb{E}^{\mathrm{o}}\left[|\mathcal{Q}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right]. \tag{6.9}$$

Note that any site in $\mathcal{Q}$ must belong to $C_{\mathbf{0}}^{\mathrm{ext}}$ in at least one percolation outside of T. In particular, given A_T^{ext}, $\mathcal{Q}$ is $\mathbb{P}^{\mathrm{o}}$-a.s. finite. Thus, arguing as in the $|T| < k$ case, we have

$$\lim_{m\to\infty} \mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{Q}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] = 0 \qquad \text{and} \qquad \lim_{m\to\infty} \mathbb{E}^{\mathrm{o}}\left[\frac{|\mathcal{R}_{k,n}(\Lambda_m)\cap\mathcal{Q}|}{m^2} \;\middle|\; A_T^{\mathrm{ext}}\right] = 0. \tag{6.10}$$

Finally, note that

$$\begin{aligned}
\mathbb{E}^{\mathrm{o}}\left[|\mathcal{R}_{k,T}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] &= \mathbb{E}\left[|\mathcal{R}_{k,T}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\cap \mathrm{O}\right] \\
&\overset{(a)}{=} \mathbb{E}\left[|C_{k,T}^{\mathrm{ext}}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\cap \mathrm{O}\right] \\
&\overset{(b)}{=} \mathbb{E}\left[|C_{k,T}^{\mathrm{ext}}\cap\Lambda_m| \;\middle|\; A_T\right],
\end{aligned}$$

where A_T is the event that $\mathbf{0}$ is in the IOC in exactly the percolations indexed by T, and $C_{k,T}^{\mathrm{ext}}$ is the set of sites of $\mathbb{Z}^2$ that belong to the IEC in at least k of the percolations indexed by T. The equality labeled (a) above is due to the fact that, conditioned on $A_T^{\mathrm{ext}}\cap O$, $\mathcal{R}_{k,T} = C_{k,T}^{\mathrm{ext}}$. The equality labeled (b) is because $A_T^{\mathrm{ext}}\cap \mathrm{O} = A_T\cap \mathrm{O}$, and moreover, the event that $\mathbf{0}$ is open in the percolations outside T is independent of the percolations indexed by T.

Thus, restricting our attention to *only* the percolations indexed by T, we can apply Corollary (6.4.3) with $n = t$ to obtain $\lim_{m\to\infty}\mathbb{E}\left[\frac{1}{m^2}|C_{k,T}^{\mathrm{ext}}\cap\Lambda_m| \;\middle|\; A_T\right] = \theta_{k,t}^{\mathrm{ext}}(p)$. Hence,

$$\lim_{m\to\infty} \mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,T}\cap\Lambda_m| \;\middle|\; A_T^{\mathrm{ext}}\right] = \theta_{k,t}^{\mathrm{ext}}(p). \tag{6.11}$$

Now using (6.5) and the fact that $\mathcal{R}_{k,T}\subset\mathcal{R}_{k,n}(\mathbb{Z}^2)$, we obtain

$$\begin{aligned}
\mathcal{R}_{k,T}\cap\Lambda_m &= \mathcal{R}_{k,T}\cap\mathcal{R}_{k,n}(\mathbb{Z}^2)\cap\Lambda_m \\
&= \mathcal{R}_{k,T}\cap\left(\mathcal{R}_{k,n}(\Lambda_m)\cup\overline{\mathcal{R}}_{k,n}(\Lambda_m)\right) \\
&= \left(\mathcal{R}_{k,T}\cap\mathcal{R}_{k,n}(\Lambda_m)\right)\cup\left(\mathcal{R}_{k,T}\cap\overline{\mathcal{R}}_{k,n}(\Lambda_m)\right),
\end{aligned}$$

in which the two sets $\mathcal{R}_{k,T}\cap\mathcal{R}_{k,n}(\Lambda_m)$ and $\mathcal{R}_{k,T}\cap\overline{\mathcal{R}}_{k,n}(\Lambda_m)$ on the RHS are disjoint (from

(6.5)). Using this, we can write the expectation term in (6.11) as follows

$$\mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,T}\cap\Lambda_m| \;\Big|\; A_T^{\mathrm{ext}}\right] = \mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,n}(\Lambda_m)\cap\mathcal{R}_{k,T}| \;\Bigg|\; A_T^{\mathrm{ext}}\right] + \mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\overline{\mathcal{R}}_{k,n}(\Lambda_m)\cap\mathcal{R}_{k,T}| \;\Bigg|\; A_T^{\mathrm{ext}}\right]. \quad (6.12)$$

Using Lemma 6.4.1, we have that

$$\lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\overline{\mathcal{R}}_{k,n}(\Lambda_m)\cap\mathcal{R}_{k,T}| \;\Bigg|\; A_T^{\mathrm{ext}}\right] \le \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{|\overline{\mathcal{R}}_{k,n}(\Lambda_m)|}{m^2} \;\Bigg|\; A_T^{\mathrm{ext}}\right] = 0 \qquad (6.13)$$

Substituting (6.12) in (6.11), and using (6.13), we get

$$\begin{aligned}
\theta_{k,t}^{\mathrm{ext}}(p) &= \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,T}\cap\Lambda_m| \;\Bigg|\; A_T^{\mathrm{ext}}\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,n}(\Lambda_m)\cap\mathcal{R}_{k,T}| \;\Bigg|\; A_T^{\mathrm{ext}}\right] \\
&\overset{\text{(a)}}{=} \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{1}{m^2}|\mathcal{R}_{k,n}(\Lambda_m)\cap\mathcal{R}_{k,T}| \;\Bigg|\; A_T^{\mathrm{ext}}\right] \\
&\qquad + \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{|\mathcal{R}_{k,n}(\Lambda_m)\cap\mathcal{Q}|}{m^2} \;\Bigg|\; A_T^{\mathrm{ext}}\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{|\mathcal{R}_{k,n}(\Lambda_m)|}{m^2} \;\Bigg|\; A_T^{\mathrm{ext}}\right],
\end{aligned} \qquad (6.14)$$

where the equality labelled (a) above is obtained using (6.10). Upon multiplying (6.7) by $\frac{1}{m^2}$, and letting $m\to\infty$, we obtain via (6.8) and (6.14):

$$\lim_{m\to\infty}\mathbb{E}^{\mathrm{o}}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] = \sum_{t=k}^{n}\sum_{\substack{T\subseteq[n]:\\|T|=t}}\theta_{k,t}^{\mathrm{ext}}(p)\,\mathbb{P}^{\mathrm{o}}(A_T^{\mathrm{ext}}).$$

Applying Proposition 6.5.1 completes the proof of the first part of the theorem. The second part of the theorem is a consequence of the proposition below. □

Proposition 6.4.4.

$$\sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^{ext}(p))^{t+j}(1-\theta^{ext}(p))^{n-j} = \mathbb{P}(Y\ge k),$$

where $Y \sim \mathrm{Bin}(n, (\theta^{ext}(p))^2)$.

Proof. Consider $Y = \sum_{i=1}^{n} X_i U_i$, where X_i, U_i, $i = 1, 2 \ldots, n$, are i.i.d. $\mathrm{Ber}(\theta^{\mathrm{ext}}(p))$ random variables. Clearly, each product $X_i U_i$ is $\mathrm{Ber}((\theta^{\mathrm{ext}}(p))^2)$, so that $Y \sim \mathrm{Bin}(n, (\theta^{\mathrm{ext}}(p))^2)$.

Alternatively, $\mathbb{P}(Y = j) = \sum_{t=0}^{n} \mathbb{P}(Y = j \mid X = t)\mathbb{P}(X = t)$, with $X = \sum_{i=1}^{n} X_i$. Thus,

$$\begin{aligned}
\mathbb{P}(Y = j) &= \sum_{t=j}^{n} \binom{t}{j} (\theta^{\mathrm{ext}}(p))^j (1 - \theta^{\mathrm{ext}}(p))^{t-j} \times \binom{n}{t} (\theta^{\mathrm{ext}}(p))^t (1 - \theta^{\mathrm{ext}}(p))^{n-t} \\
&= \sum_{t=j}^{n} \binom{n}{t} \binom{t}{j} (\theta^{\mathrm{ext}}(p))^{t+j} (1 - \theta^{\mathrm{ext}}(p))^{n-j}.
\end{aligned}$$

Hence,

$$\mathbb{P}(Y \geq k) = \sum_{j=k}^{n} \sum_{t=j}^{n} \binom{n}{t} \binom{t}{j} (\theta^{\mathrm{ext}}(p))^{t+j} (1 - \theta^{\mathrm{ext}}(p))^{n-j},$$

from which, upon exchanging the order of the summations, we get the expression in the statement of the proposition. $\square$

6.5 Proof of key lemmas

In this section, we collect the proofs of Lemma 6.4.1 and Corollary 6.4.2 which were used in the proof of Theorem 6.3.1. We remark here that similar ideas are used to prove analogous results on random geometric graphs as well.

Lemma 6.4.1. *For* $p > p_c$, *we have*

$$\lim_{m \to \infty} \frac{1}{m^2} \, \mathbb{E}^{\mathrm{o}} \left[|\overline{\mathcal{R}}_{k,n}(\Lambda_m)| \right] = 0$$

Proof. Fix an $\epsilon > 0$. We will find an m_0 such that $\frac{1}{m^2} \mathbb{E}^{\mathrm{o}} \left[\overline{\mathcal{R}}_{k,n}(\Lambda_m) \right] < \epsilon$ for all $m \geq m_0$. This will prove the lemma.

Any node in $\overline{\mathcal{R}}_{k,n}(\Lambda_m)$ has a conduit in at least k out of the n packet transmissions on $\mathbb{Z}^2$ and receives at least one packet from outside Λ_m. Denote by M_j the event that node j receives at least one of the n packets from outside Λ_m. Recall that this means that node

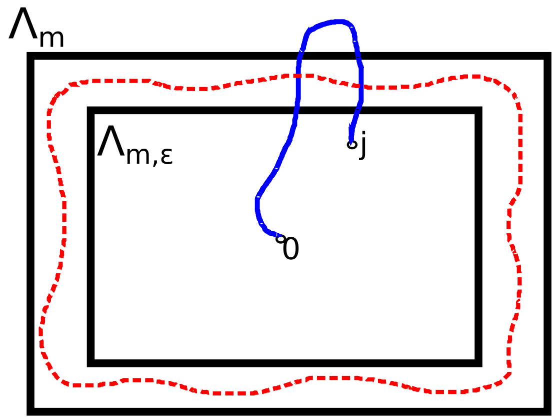


Figure 6.9: Illustration of open loop in the annulus $\Lambda_m \backslash \Lambda_{m,\epsilon}$. Here the vertex j receives the packet from origin 0, only along the path that is depicted.

j does not have any Λ_m-conduit for this packet. We then have,

$$\begin{aligned}\mathbb{E}^o\left[\frac{|\overline{\mathcal{R}}_{k,n}(\Lambda_m)|}{m^2}\right] &\leq \mathbb{E}^o\left[\frac{1}{m^2}\sum_{j\in\Lambda_m}\mathbb{1}_{M_j}\right], \\ &= \mathbb{E}^o\left[\frac{1}{m^2}\sum_{j\in\Lambda_{m,\epsilon}}\mathbb{1}_{M_j}\right] + \mathbb{E}^o\left[\frac{1}{m^2}\sum_{j\in\Lambda_m\backslash\Lambda_{m,\epsilon}}\mathbb{1}_{M_j}\right],\end{aligned}$$

where $\mathbb{1}_{M_j}$ is the indicator random variable for the event M_j, i.e., $\mathbb{1}_{M_j} = 1$ if M_j occurs, and $\mathbb{1}_{M_j} = 0$ otherwise. Since there are $m^2 - m^2\left(1 - \frac{\epsilon}{4}\right) = \frac{m^2\epsilon}{4}$ nodes in $\Lambda_m \backslash \Lambda_{m,\epsilon}$, the latter term can be further bounded to obtain,

$$\mathbb{E}^o\left[\frac{|\overline{\mathcal{R}}_{k,n}(\Lambda_m)|}{m^2}\right] \leq \frac{1}{m^2}\sum_{j\in\Lambda_{m,\epsilon}}\mathbb{P}^o\left(M_j\right) + \frac{\epsilon}{4}. \tag{6.15}$$

The summation above can be split over those nodes which are on the boundary of $\Lambda_{m,\epsilon}$ and those in the interior. The former term contains at most $4m\sqrt{1-\epsilon/4}$ nodes. The latter term involves those nodes which receive at least one packet from outside Λ_m. Hence, in at least one percolation, such nodes have a path from the origin as shown in Fig. 6.9. This, then implies that there cannot be an open loop in the annulus $\Lambda_m \backslash \Lambda_{m,\epsilon}$ as indicated by the dotted line in Fig. 6.9. Let K_m be the event that there is no open loop around the origin in the annulus $\Lambda_m \backslash \Lambda_{m,\epsilon}$ in at least one percolation. We then obtain,

$$\begin{aligned}\frac{1}{m^2}\sum_{j\in\Lambda_{m,\epsilon}}\mathbb{P}^{\mathrm{o}}\left(M_j\right) &\leq \frac{1}{m^2}\left[4m\sqrt{1-\frac{\epsilon}{4}}\,\right]+\left(1-\frac{\epsilon}{4}\right)\mathbb{P}^{\mathrm{o}}\left(K_m\right)\\ &= \frac{4}{m}\sqrt{1-\frac{\epsilon}{4}}+\left(1-\frac{\epsilon}{4}\right)\left(1-\mathbb{P}^{\mathrm{o}}\left(K_m^c\right)\right). \end{aligned} \tag{6.16}$$

The event K_m^c is the event that there is an open loop in the annulus $\Lambda_m\backslash\Lambda_{m,\epsilon}$ in each of the n percolations. Note that this is an increasing event and so is the event O. Using the FKG inequality (Appendix A.3), we have that

$$\begin{aligned}\mathbb{P}^{\mathrm{o}}\left(K_m^c\right) &= \frac{\mathbb{P}\left(K_m^c\cap \mathrm{O}\right)}{\mathbb{P}\left(\mathrm{O}\right)},\\ &\overset{(FKG)}{\geq} \frac{\mathbb{P}\left(K_m^c\right)\mathbb{P}\left(\mathrm{O}\right)}{\mathbb{P}\left(\mathrm{O}\right)},\\ &= \mathbb{P}\left(K_m^c\right). \end{aligned} \tag{6.17}$$

On $\{0,1\}^{\mathbb{Z}^2}$, define Ann to be the event that there is an open loop in the annulus $\Lambda_m\backslash\Lambda_{m,\epsilon}$. Exploiting the independence of packet transmissions, we have that $\mathbb{P}\left(K_m^c\right)=\mathbb{P}_1\left(\mathsf{Ann}\right)^n$. Substituting (6.17) and (6.16) in (6.15), and using this result, we obtain,

$$\mathbb{E}^{\mathrm{o}}\left[\frac{|\overline{\mathcal{R}}_{k,n}(\Lambda_m)|}{m^2}\right]\leq\frac{4}{m}\sqrt{1-\frac{\epsilon}{4}}+\left(1-\frac{\epsilon}{4}\right)\left(1-\mathbb{P}_1\left(\mathsf{Ann}\right)^n\right)+\frac{\epsilon}{4}.$$

For super-critical site percolation process on $\mathbb{Z}^2$ and a fixed $\epsilon>0$, the probability of an open loop in the annulus $\Lambda_m\backslash\Lambda_{m,\epsilon}$ around the origin is known to approach 1 as $m\to\infty$ (see [80] for an idea of the proof, and [81] for specific results for site percolation) i.e. $\mathbb{P}_1(\mathsf{Ann})\to 1$ as $m\to\infty$. Thus we can find an m_0 such that each of the first two terms on the RHS in the above expression are less than $\frac{\epsilon}{4}$ for all $m\geq m_0$. This is the required m_0. □

Corollary 6.4.2. *For $p>p_c$, we have*

$$\lim_{m\to\infty}\frac{1}{m^2}\mathbb{E}^{\mathrm{o}}\left[\overline{\mathcal{R}}_{k,n}(\Lambda_m)\middle|A_T^{ext}\right]=0.$$

Proof. The proof is along similar lines as that of Lemma 6.4.1 but with additional conditioning on the event A_T^{ext}. More specifically, (6.16) would have $\mathbb{P}^{\text{o}}\left\{K_m^c \mid A_T^{\text{ext}}\right\}$ on the RHS. Notice that A_T^{ext} is an increasing event and hence $\mathrm{O} \cap A_T^{\text{ext}}$ is also increasing. Thus,

$$\begin{aligned}\mathbb{P}^{\text{o}}\left(K_m^c \mid A_T^{\text{ext}}\right) &= \frac{\mathbb{P}\left(K_m^c \cap A_T^{\text{ext}} \cap \mathrm{O}\right)}{\mathbb{P}\left(A_T^{\text{ext}} \cap \mathrm{O}\right)} \\ &\overset{(FKG)}{\geq} \frac{\mathbb{P}\left(K_m^c\right)\mathbb{P}\left(A_T^{\text{ext}} \cap \mathrm{O}\right)}{\mathbb{P}\left(A_T^{\text{ext}} \cap \mathrm{O}\right)} \\ &= \mathbb{P}\left(K_m^c\right). \end{aligned} \tag{6.18}$$

Using this in (6.16) and following subsequent steps from the lemma, we get the statement of the corollary. □

It is to be justified that such conditioning can indeed be done, i.e., the event A_T^{ext} has a positive probability for the specified range of values of p. The following proposition relates the probability of the event A_T^{ext}, conditioned on the event that the origin is open in all n percolations, to $\theta^{\text{ext}}(p)$.

Proposition 6.5.1. *For any $T \subseteq [n]$ with $|T| = t$, we have*

$$\mathbb{P}^{\text{o}}(A_T^{ext}) \;=\; (\theta^{ext}(p))^t(1-\theta^{ext}(p))^{n-t}.$$

Proof. By definition, $\mathbb{P}^{\text{o}}(A_T^{\text{ext}}) = \mathbb{P}(A_T^{\text{ext}} \mid \mathrm{O})$. Note that, in a given percolation, conditioned on $\mathbf{0}$ being open, the event $\{\mathbf{0}$ is in the IEC$\}$ is the same as the event $\{\mathbf{0}$ is in the IOC$\}$. Consequently, conditioned on O, the event A_T^{ext} is the same as the event, A_T, that the origin is in the IOC in exactly the percolations indexed by T. Hence,

$$\mathbb{P}^{\text{o}}(A_T^{\text{ext}}) = \mathbb{P}(A_T \mid \mathrm{O}) = \frac{\mathbb{P}(A_T \cap \mathrm{O})}{\mathbb{P}(\mathrm{O})}.$$

The denominator equals p^n. The numerator is the event that the origin is in the IOC in exactly the percolations indexed by T, and is open but in a finite cluster in the remaining $n - |T|$ percolations. In a given percolation, the probability that the origin is open but in a finite cluster is $p - \theta(p)$. Thus, we have $\mathbb{P}(A_T \cap \mathrm{O}) = (\theta(p))^{|T|}(p - \theta(p))^{n-|T|}$. The result now follows from the fact (Lemma 6.2.1) that $\theta^{\text{ext}}(p) = \frac{\theta(p)}{p}$. $\square$

Since $\theta^{\text{ext}}(p) > 0$ for $p > p_c$, we have that $\mathbb{P}^{\mathrm{o}}(A_T^{\text{ext}}) > 0$ as well.

6.6 Discussion

In this section, we give justifications and heuristics for some of the assumptions made in our analysis.

6.6.1 Super-critical region

Our entire analysis for grids is based on the assumption (Assumption 1) that we operate in the super-critical region for the site-percolation process. We give an explanation for the same here. Recall that we want values of the forwarding probability p for which the expected fraction of successful receivers, $\mathbb{E}[\frac{1}{m^2}R_{k,n}(\Lambda_m)]$ is at least $1-\delta$, for some (small) $\delta > 0$. Hence, we need $\mathbb{E}[\frac{1}{m^2}|\mathcal{R}_{k,n}(\mathbb{Z}^2) \cap \Lambda_m|] \geq 1 - \delta$. If we would like this to hold for all sufficiently large m, then p must be such that $\mathcal{R}_{k,n}(\mathbb{Z}^2)$ has infinite cardinality. This implies, due to the correspondence between probabilistic forwarding and site percolation on $\mathbb{Z}^2$, that p must be such that there exists an infinite (open/extended) cluster in the site percolation process. Thus, we must operate in the super-critical region $p > p_c$. It can also be seen from the simulation results in Figs. 6.5 and 6.6 that $\tau_{k,n,\delta}$ is minimized when $p_{k,n,\delta}$ is in the super-critical region. Further, from Fig. 6.8(a), which provides the minimum forwarding probability obtained numerically from (6.3), and which is used to generate the plots in Fig. 6.8(b), it is clear that the expected total number of transmissions is indeed minimized when operating in the super-critical region. We use these arguments as justification for considering only the $p > p_c$ case in our analysis.

6.6.2 Insufficiently large m

We now re-visit the disparity seen in Fig. 6.6 between the $\tau_{k,n,\delta}$ curves (normalized by the grid size m^2) for Λ_{31} and Λ_{501} obtained via simulations, and the corresponding curve for large grids Λ_m obtained via (6.4). As discussed previously, the numerical evaluation of the RHS of (6.4) relies on the approximation to $p_{k,n,\delta}$ in (6.3), which, for fixed k, n and δ, is valid only for sufficiently large m. In the regime where the approximation is not valid (as happens for $n \geq 130$ and $m = 501$ in Fig. 6.6), there is a small discrepancy between the true value of $p_{k,n,\delta}(\Lambda_m)$ obtained via simulations, and the approximation in (6.3). While this discrepancy is too small to be seen in the plots in Fig. 6.5, it gets blown up when evaluating $\tau_{k,n,\delta}$ using the expression in (6.4), which involves $\theta^{\text{ext}}(p)$. This blow-up is attributable to the fact that $\theta^{\text{ext}}(p)$ exhibits a sharp phase transition around $p = 0.6$ (see Fig. 6.4), so that small changes in p near 0.6 translate to large changes in $\theta^{\text{ext}}(p)$.

Interestingly, our simulations also indicate that for *any* value of m, the true curve for $\frac{1}{m^2}\tau_{k,n,\delta}(\Lambda_m)$ always lies on or above the curve for the "large-Λ_m approximation" obtained via (6.4) and (6.3). We attempt an explanation for this here. We conjecture that the large-m approximation in (6.3) is in fact an inequality valid for all m, at least when δ is small.

Conjecture 6.6.1. *Fix $\delta \in (0, 1/8)$. Then, for any k, n and m, we have*

$$p_{k,n,\delta}(\Lambda_m) \;\geq\; \inf\{p \mid Pr(Y \geq k) \geq 1 - \delta\}, \tag{6.19}$$

where $Y \sim \mathrm{Bin}(n, (\theta^{ext}(p))^2)$.

Thus, assuming the validity of the conjecture, the expected total number of transmissions, $\tau_{k,n,\delta}(\Lambda_m)$, at a forwarding probability equal to $p_{k,n,\delta}(\Lambda_m)$ is at least as large as that when the forwarding probability is set to be equal to the RHS of (6.3) (or (6.19)). We next provide an argument in support of the conjecture.

Recall that

$$p_{k,n,\delta}(\Lambda_m) = \inf\left\{p \;\middle|\; \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \geq 1 - \delta\right\},$$

while the RHS of (6.19) is, by virtue of Theorem 6.3.1,

$$\inf\left\{p \;\middle|\; \lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \geq 1 - \delta\right\}.$$

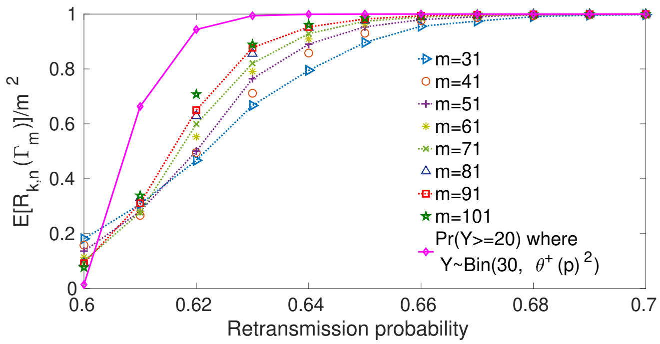


Figure 6.10: Plot of the expected fraction of nodes that receive at least $k = 20$ out of $n = 30$ packets in a 501×501 grid. Expectation over 100 iterations.

Thus, it would suffice to show that when p is large enough to ensure that $\mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \geq 1-\delta$, we also have $\lim_{m\to\infty} \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right] \geq \mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right]$. This seems to be true: simulation results (see Fig. 6.10) in fact indicate that, for fixed k and n, and p sufficiently above criticality, $\mathbb{E}\left[\frac{R_{k,n}(\Lambda_m)}{m^2}\right]$ is an increasing function of m.

The intuition behind the increasing nature of the fraction of receivers can be illustrated via the case of $k = 1$ and $n = 1$. Consider a node v on the boundary of Λ_m which receives the sole packet from outside Λ_m. Let us further suppose that the path through which it receives the packet is contained within Λ_{m+l} for some small $l > 0$. Node v is not a successful receiver in Λ_m but it is successful in Λ_{m+l}. Additionally, nodes in the Λ_{m+l}-conduit of v (and the neighbours of these nodes) that are not successful receivers in Λ_m become successful receivers in Λ_{m+l}. Moreover, if node v transmits the packet, there are additional nodes in the interior of Λ_m that receive the packet. So, increasing the grid size from m to $m+l$ not only leads to an increase in the number of receivers on the boundary but also results in additional receivers in the bulk. This suggests that the expected number of receivers in Λ_m increases in chunks of m^2 rather than just m. Unfortunately, a rigorous proof of this fact eludes us.

6.6.3 Tree vs. Grid

The analysis on the binary tree and the grid reveals that there is a significant benefit to introducing coding-based redundancy into the probabilistic forwarding protocol when the underlying network topology is well-connected (as in a large grid), but not so when the underlying network is a tree. The benefit is in terms of a reduction in the overall number of transmissions needed for a successful broadcast. Here, we give a qualitative explanation for this behaviour.

Recall that a "near-broadcast" is when the expected fraction of successful receivers is at least $1 - \delta$, for some small $\delta > 0$. On a binary tree, leaves constitute (approximately) 50% of the nodes. So, for a near-broadcast on a binary tree, the expected fraction of successful receivers among the leaf nodes should be at least $1 - 2\delta$. It then follows, via linearity of expectation, that the probability of a given leaf node receiving at least k of the n coded packets should be at least $1 - 2\delta$. Since there is a unique path from the source (root node) to a leaf node on the tree, for a leaf node to receive a packet, every node on this unique path needs to transmit the packet. Hence, for a tree with a large height H, to ensure a near-broadcast, the forwarding probability needed for a leaf node to receive (with high probability) at least k out of n packets must necessarily be high. Of course, as Lemma 4.3.1 shows, the minimum forwarding probability, $p_{k,n,\delta}$, needed for a near-broadcast decreases to 0 monotonically in n. However, the estimates in Section 5.2 show that, for a binary tree, $p_{k,n,\delta}$ does not decrease quickly enough in n to offset the increase in the number n of packets to be transmitted, resulting in a net overall increase in the expected total number of transmissions as n increases.

On the other hand, on a grid, there are multiple paths from the source to any node. A packet is received by a node if all the nodes on at least one of these paths transmit it. It is the existence of these multiple paths between the source node and any other node on the grid that causes the minimum forwarding probability $p_{k,n,\delta}$, for fixed k, δ and increasing values of n, to decrease sharply at first (as seen in Fig. 6.5), which results in an initial decrease in the expected total number of transmissions. The effect of multiple paths is not so strong after a point, and addition of coded packets does not impact $p_{k,n,\delta}$ much. This causes a slowdown in the rate of decrease of the minimum forwarding probability in n, which then results in an increase in the expected total number of transmissions.

6.7 Conclusions

The introduction of coded packets along with the probabilistic forwarding mechanism is beneficial on the square grid. This benefit is in terms of the expected total number of transmissions required for a near-broadcast.

The analysis of the mechanism extends to other lattice structures as well. A careful scrutiny of the analysis reveals ergodic theorems to be the main workhorses behind the proofs and the numerical results. Regular lattice structures, such as the triangular or the hexagonal grid, also admit such ergodic theorems. The expressions for the fraction of successful receivers in Theorem 6.3.1 and the expected total number of transmissions in (6.4) hold as is, but with the percolation probability of the lattice considered.

In order to obtain numerical results, an estimate of the percolation probability needs to be obtained. While the percolation thresholds are known for some of the graphs (for e.g., $p_c(\text{triangular}) = \frac{1}{2}$), there are no known analytical expressions for $\theta(p)$. The ergodic theorems can be used to obtain an estimate of $\theta(p)$ similar to the procedure followed for the square grid here.

The analysis presented here suffices to justify that introducing coded packets along with probabilistic forwarding makes it more energy-efficient. However, it does not characterize the exact number of coded packets which minimizes the total number of transmissions. Quantifying this optimal number of coded packets is a possible direction for future work.

Part III

Random graphs

Chapter 7

Random Geometric Graphs

Modelling and analysis of the probabilistic forwarding protocol on random graphs brings to the fore multiple new challenges. Firstly, the randomness in the graph could arise due to the number of nodes or the way the graph is constructed (or both). Secondly, there could be multiple ways in which a source node is selected from the random graph to initiate the broadcasts. Lastly, the probabilistic forwarding mechanism needs a connected underlying graph. Depending on the random graph model, this entails either working only on the connected realizations of the graph or restricting to a connected subgraph of the random graph. These have to be accounted for while analyzing the probabilistic forwarding mechanism.

Ad-hoc networks are distributed networks with no centralized infrastructure. Applications involving the Internet of Things (IoT), such as healthcare, smart factories and homes, intelligent transport etc., have lead to wide-spread presence of dense ad-hoc networks. Individual nodes in these networks are typically low-cost and energy-constrained, having limited computational ability and knowledge of the network topology

Random network models have found wide acceptance in modeling wireless ad-hoc networks. In particular, random geometric graphs (RGGs) have been used in the literature to model spatially distributed networks (see e.g. [84] and [85]). These are generated by scattering (a Poisson number of) nodes in a finite area uniformly at random and connecting nodes within a pre-specified distance. The random distribution of nodes captures the variability in the deployment of the nodes of an ad-hoc network. The distance threshold conforms to the maximum range at which a transmission from a node, with maximum power, is received reliably. A more formal description of our network setting is provided in the next section.

In this thesis, our primary interest will be on RGGs since they aptly capture macro-phenomenon on deployments of ad-hoc networks. While the approach for analyzing the mechanism is similar to that of grids, the additional intricacies brought about due to the challenges discussed above make it more interesting.

7.1 Problem formulation

We begin by describing our setting for the specific case of random geometric graphs. While some of the material here was presented in Chapter 3, additional notation specific to RGGs is introduced here.

7.1.1 Network setup

A random geometric graph is parametrized by the intensity λ and the distance threshold r. It suffices to study them by keeping one of the parameters fixed. In our treatment, we will fix the distance parameter r to be equal to 1, and study various properties as a function of the intensity, λ.

Construct a random geometric graph G_m with intensity λ and distance threshold $r = 1$ on $\Gamma_m := \left[\frac{-m}{2}, \frac{m}{2}\right]^2$ as follows:

- **Step 1:** Sample the number of points, N, from a Poisson distribution with mean $\lambda\nu(\Gamma_m)$. Here, $\nu(\cdot)$ is the Lebesgue measure on $\mathbb{R}^2$. Therefore, $N \sim \text{Poi}(\lambda m^2)$.

- **Step 2:** Choose points $X_1, X_2, \cdots, X_N$ uniformly and independently from Γ_m. These form the points of a Poisson point process (see [73, Section 2.5]) Φ, and constitute the vertex set of G_m.

- **Step 3:** Place an edge between any two vertices which are within Euclidean distance $r = 1$ of each other.

To carry out probabilistic forwarding over G_m, we need to fix a source. For this, we will assume that there is a point at the origin $\mathbf{0} = (0, 0) \in \mathbb{R}^2$. More specifically, a graph $G_m^{\mathbf{0}}$ is created with the underlying point process $\Phi^{\mathbf{0}} \triangleq \Phi \cup \{\mathbf{0}\}$, as the vertex set and introducing additional edges from $\mathbf{0}$ to nodes which are within $B_1(\mathbf{0})$, to the edge set of G_m. Here, $B_1(\mathbf{0})$ (more generally, $B_1(\mathbf{v})$ for $\mathbf{v} \in \mathbb{R}^2$) is a closed Euclidean ball of radius 1 centered at $\mathbf{0}$ (resp. $\mathbf{v}$).

The inclusion of an additional point at the origin $\mathbf{0}$ means that all the probabilistic computations need to be made with respect to the Palm probability given a point at the origin. We direct the reader to [86, Ch. 1.4] for an in-depth treatment of Palm theory. Physically, the Palm probability must be interpreted as the probability conditional on the event that the origin is a point of the point process. We denote the Palm probability by $\mathbb{P}^{\mathbf{0}}$ and the expectation with respect to it by $\mathbb{E}^{\mathbf{0}}$.

The origin here is a distinguished vertex. Broadcasts initiated from it can be received by the nodes which are present in the component of the origin only. Denote by $C_{\mathbf{0}} \equiv C_{\mathbf{0}}(G_m^{\mathbf{0}})$, the set of nodes in the component of the origin in $G_m^{\mathbf{0}}$. The component of the origin in $G_m^{\mathbf{0}}$ forms the underlying connected graph, which we denote by G.

7.1.2 Problem definition

Equipped with the underlying network, G, the probabilistic forwarding algorithm proceeds as in Chapter 2, with the source at the origin $\mathbf{0}$. Denote by $R_{k,n}(G)$, the number of nodes that receive at least k out of the n coded packets. These are the *successful receivers*. We sometimes denote this by $R_{k,n}(G_m^{\mathbf{0}})$ to explicitly bring out the dependence on m. Given a $\delta > 0$, we are interested in the minimum forwarding probability p, such that the expected fraction of successful receivers is at least $1 - \delta$. The expectation here is over the probabilistic forwarding protocol for a fixed realization of G. In reality, the proposed broadcasting algorithm of probabilistic forwarding with coded packets, should give a good performance for any realization of the underlying graph. In other words, we would want the expected fraction of successful receivers to be at least $1-\delta$, for every realization of G. However, in our formulation we relax this condition by asking for it only in an expected sense. More specifically, we define

$$p_{k,n,\delta} = \inf \left\{ p \;\middle|\; \mathbb{E}\left[\frac{R_{k,n}(G_m^{\mathbf{0}})}{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|}\right] \geq 1-\delta \right\}, \tag{7.1}$$

where the expectation is over both the graph G as well as the probabilistic forwarding mechanism. Note that, from our construction, $R_{k,n}(G) = R_{k,n}(G_m^{\mathbf{0}}) \subseteq C_{\mathbf{0}}(G_m^{\mathbf{0}})$. The number of successful receivers is normalized by the total number of vertices in G, which is the same as the number of vertices within the component of the origin, $C_{\mathbf{0}}(G_m^{\mathbf{0}})$.

The performance measure of interest, denoted by $\tau_{k,n,\delta}$, is the expected total number of transmissions across all nodes in $C_{\mathbf{0}}$, when the forwarding probability is set to $p_{k,n,\delta}$. Here again the expectation is over both G and the probabilistic forwarding protocol. In subsequent sections, we will cast the probabilistic forwarding mechanism as a marked point process and use results from ergodic theory to obtain the expected value of the number of successful receivers and the overall number of transmissions.

7.2 Preliminaries

In this section, we introduce the tools required to characterize the performance of the probabilistic forwarding algorithm. Our analysis proceeds by relating the mechanism on the finite graph G, and the RGG constructed on the whole $\mathbb{R}^2$ plane. The probabilistic forwarding mechanism on the RGG is modeled using marked point processes which are detailed here.

7.2.1 Random geometric graphs on $\mathbb{R}^2$

A *point process* on $\mathbb{R}^2$ is a random collection of finite or countably infinite points with no accumulation points. It is often easier to think of a point process as a counting measure $\Phi := \sum_i \varepsilon_{X_i}$, where ε_x is the *Dirac* measure; for $A \subset \mathbb{R}^2$, $\varepsilon_x(A) = 1$ if $x \in A$ and $\varepsilon_x(A) = 0$ if $x \notin A$. Consequently, $\Phi(A)$ gives the number of points within A.

Definition 1. *(see [73, Section 2.5]) A homogeneous Poisson point process (PPP), Φ, of intensity $\lambda > 0$ is a random set of points in $\mathbb{R}^2$ which satisfy the following conditions:*

- *For mutually disjoint regions of $\mathbb{R}^2$, $A_1, A_2, \cdots, A_r$, the random variables denoting the number of points in each of those regions, $\Phi(A_1), \Phi(A_2), \cdots, \Phi(A_r)$, are mutually independent.*
- *For any bounded $A \in \mathcal{B}(\mathbb{R}^2)$, $\mathbb{P}(\Phi(A) = l) = \dfrac{e^{-\lambda\nu(A)}(\lambda\nu(A))^l}{l!}$, where $\nu(A)$ denotes the Lebesgue measure of A.*

If λ is a (non-constant) function of $x \in \mathbb{R}^2$, then Φ is an inhomogeneous Poisson point process.

Definition 2. *A random geometric graph, $RGG(\Phi, r)$ on $\mathbb{R}^2$, is a graph constructed with an underlying Poisson point process Φ as the vertex set. The edge set of the graph includes all the edges between any two points of Φ which are within Euclidean distance r of each other. When the underlying Poisson point process Φ is homogeneous, we denote the random geometric graph by $RGG(\lambda, r)$. We will also use the notation $\mathcal{G} \sim RGG(\lambda, r)$ to indicate that the graph $\mathcal{G}$ is a random geometric graph with intensity λ and distance threshold r.*

The RGG model was first introduced by Gilbert in 1961 to study wireless networks and is also called the Gilbert disc model. It is known that $RGG(\lambda_1, r_1)$ and $RGG(\lambda_2, r_2)$ share similar connectivity properties if $\lambda_1 r_1^2 = \lambda_2 r_2^2$ (see [84, Chapter 7.3]). As in the finite case, we will always take $r = 1$, and study different properties as a function of λ.

Our approach to analyzing the probabilistic forwarding mechanism on G is to relate it to the probabilistic forwarding mechanism on a RGG generated on the whole $\mathbb{R}^2$ plane with the origin as the source. This means that the vertex set of the RGG is a Poisson point process, Φ, on $\mathbb{R}^2$. We refer the reader to [85] or [86] for the background needed on Poisson point processes. In particular, we use the procedure outlined in [86, Section 1.3] to construct the RGG on the whole $\mathbb{R}^2$ plane.

Create a tiling of the $\mathbb{R}^2$ plane with translations of Γ_m, i.e., $\Gamma_{i,j} := (im, jm) + \Gamma_m$ for $i, j \in \mathbb{Z}$. On each such translation, $\Gamma_{i,j}$, construct an independent copy of a Poisson point process with intensity λ as described in steps 1 and 2 of Section 7.1.1. The random geometric graph ($\mathcal{G}$) is constructed by connecting vertices which are within distance 1 of each other. We then say $\mathcal{G} \sim RGG(\lambda, 1)$.

It is known that the $RGG(\lambda, 1)$ model on $\mathbb{R}^2$ shows a phase transition phenomenon (see e.g. [87]). For $\lambda > \lambda_c$, the *critical intensity*, there exists a unique infinite cluster, $C \equiv C(\Phi)$, in the RGG almost surely. The value of λ_c is not exactly known, but simulation studies such as [88] indicate that $\lambda_c \approx 1.44$. The *percolation probability* $\theta(\lambda)$ is defined as the probability that the origin is present in the infinite cluster C, i.e., $\theta(\lambda) := \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C)$. We remark here that there is no known analytical expression for $\theta(\lambda)$ nor are there good approximations. Since we are interested in large dense networks, we will assume throughout our analysis that we operate in the super-critical region, i.e., $\lambda > \lambda_c$.

7.2.2 Marked Point Process

During the course of the probabilistic forwarding protocol on the RGG, each node decides independently whether to forward a particular packet with probability p. Marked point processes (MPPs) turn out to be a natural way to model such functions of an underlying point process.

Definition 3. *Let $\Phi = \sum_i \varepsilon_{X_i}$ be a Poisson point process on $\mathbb{R}^2$. With each point X_i of Φ, associate a mark Z_i taking values in some measurable space $(\mathbb{K}, \mathcal{K})$ such that $\{Z_i\}_{i \in \mathbb{N}} \overset{iid}{\sim} \Pi(\cdot)$. Then, $\tilde{\Phi} = \sum_i \varepsilon_{(X_i, Z_i)}$ is called an* iid marked point process *on $\mathbb{R}^2 \times \mathbb{K}$ with* mark distribution $\Pi(\cdot)$.

We now state an ergodic theorem for MPPs which is used to obtain some key results required in the analysis of the probabilistic forwarding protocol in Section 7.3.

7.2.3 Ergodic theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space over which an iid marked point process $\tilde{\Phi} = \sum_i \varepsilon_{(X_i, Z_i)}$ is defined with mark distribution $\Pi(\cdot)$. Let $\theta_x : \Omega \to \Omega$, for $x \in \mathbb{R}^2$, be the operator which shifts each point of $\tilde{\Phi}$ by $-x$, i.e., $\theta_x \tilde{\Phi} = \sum_i \varepsilon_{(X_i - x, Z_i)}$ and let $(\mathbb{K}, \mathcal{K})$

be the measurable space of marks. Let $f : \mathbb{K} \times \Omega \to \mathbb{R}_+$ be a non-negative function of the MPP. Then, by the ergodic theorem for marked random measures (see [89, Theorem 8.4.4]), we have

$$\frac{1}{\nu(\Gamma_m)} \sum_{X_i \in \Gamma_m} f(Z_i, \theta_{X_i}(\omega)) \to \lambda \int_{\mathbb{K}} \mathbb{E}^{(\mathbf{0},z)} \left[f(z,\omega) \right] \Pi(dz) \qquad \mathbb{P}\text{-a.s.} \tag{7.2}$$

as $m \to \infty$, where $\mathbb{E}^{(\mathbf{0},z)}$ is the expectation with respect to the Palm probability $\mathbb{P}^{(\mathbf{0},z)}$ conditional on the mark, z. If $f(z,\omega) = f(\omega)$, then (7.2) reduces to

$$\frac{1}{\nu(\Gamma_m)} \sum_{X_i \in \Gamma_m} f(\theta_{X_i}(\omega)) \overset{m\to\infty}{\longrightarrow} \lambda \mathbb{E}^{\mathbf{0}} \left[f(\omega) \right] \qquad \mathbb{P}\text{-a.s..} \tag{7.3}$$

7.3 Probabilistic forwarding and MPPs

In this section, we formulate probabilistic forwarding mechanism using the framework of marked point processes. Ergodic theorems for MPPs are then used to derive relevant results which will be used to obtain estimates for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$. It should be noted here that all the graphs and point processes discussed in this section are on the whole $\mathbb{R}^2$ plane.

7.3.1 Single packet probabilistic forwarding

Consider the probabilistic forwarding of a single packet on $\mathcal{G} \sim RGG(\Phi, 1)$ defined on a PPP Φ of intensity λ on $\mathbb{R}^2$. Let $\mathcal{G}^{\mathbf{0}}$ be the graph created with the underlying point process being $\Phi^{\mathbf{0}} \triangleq \Phi \cup \{\mathbf{0}\}$ as the vertex set, and introducing additional edges from $\mathbf{0}$ to nodes which are within $B_1(\mathbf{0})$, to the edge set of $\mathcal{G}$. We assign a mark 1 to a node if it decides to transmit the packet and 0 otherwise. Thus, the mark space is $\mathbb{K} = \{0, 1\}$ and $\tilde{\Phi}$ is an iid MPP with a $Ber(p)$ mark distribution. Note that the origin, $\mathbf{0}$, has mark 1 since it always transmits the packet. Also, the subset of nodes which have mark 1 form a thinned point process of intensity λp, and the subset of vertices with mark 0 form a $\lambda(1-p)$–thinned process. Denote these by Φ^+ and Φ^- respectively, and the corresponding

RGGs by $\mathcal{G}^+$ and $\mathcal{G}^-$. Notice that the set of vertices of Φ^+ which are in the same cluster as the origin are the vertices which receive the packet from the source and transmit it. Thus, the number of vertices in the cluster containing the origin in $\mathcal{G}^+$ (call this set of nodes $C_{\mathbf{0}}^+$), is the number of transmissions of the packet.

In addition to the nodes of the cluster containing the origin in $\mathcal{G}^+$, the nodes of $\mathcal{G}^-$ which are within distance 1 from them, also receive the packet. To account for them, we define for any cluster of nodes $S \subset \Phi^+$, the *boundary* of S as

$$\partial S = \{\mathbf{v} \in \Phi^- | B_1(\mathbf{v}) \cap S \neq \emptyset\},$$

and the *extended cluster* of S to be $S^{\text{ext}} = S \cup \partial S$. Then, the receivers are the nodes in $C_{\mathbf{0}}^{\text{ext}}$. We refer to this as the extended cluster of the origin.

Our interest is in large and dense networks in which the origin is likely to be in the infinite cluster of $\mathcal{G}^{\mathbf{0}}$. Moreover, since we are interested in a large fraction of nodes in the network to be successful receivers, the extended cluster of the origin has to comprise of a significant number of nodes within Γ_m. In the limit of large m, this means that the extended cluster of the origin is the *infinite extended cluster* (IEC), C^{ext}, defined as the extended cluster of $C^+ := C(\Phi^+)$. This also means that the transmitters correspond to the nodes within Γ_m of the infinite cluster of Φ^+, C^+. Thus, in the thermodynamic limit, the expected number of vertices in $C_{\mathbf{0}} \cap \Gamma_m$ (resp. $C_{\mathbf{0}}^{\text{ext}} \cap \Gamma_m$) is well-approximated by the expected number of vertices within Γ_m of the infinite cluster C^+ (resp., of the IEC C^{ext}) for large m. We use the ergodic theorem stated in Section 7.2.3 to obtain almost sure results for the fraction of nodes within Γ_m of the infinite cluster C^+ and the IEC C^{ext} in terms of the percolation probability $\theta(\lambda)$.

7.3.2 Application of the ergodic theorem

Specializing the statement in (7.2) to the probabilistic forwarding of a single packet where $\mathbb{K} = \{0, 1\}$ and the marks are independent, conditional on Φ, with distribution given by

$\Pi(1) = 1 - \Pi(0) = p$, we obtain,

$$\frac{1}{\nu(\Gamma_m)} \sum_{X_i \in \Gamma_m} f(Z_i, \theta_{X_i}(\omega)) \overset{m\to\infty}{\longrightarrow} \lambda p \, \mathbb{E}^{(\mathbf{0},1)}[f(1,\omega)] + \lambda(1-p) \, \mathbb{E}^{(\mathbf{0},0)}[f(0,\omega)] \qquad \mathbb{P}\text{-a.s.}. \tag{7.4}$$

We will now use (7.3) and (7.4) to obtain key results which will be used to analyze the probabilistic forwarding of a single packet on $\mathbb{R}^2$. In particular, we substitute different functions f in (7.3) and (7.4) to obtain the following results:

- $f(z,\omega) = 1$. The ergodic theorem in (7.3) results in

$$\frac{\Phi(\Gamma_m)}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda \qquad \mathbb{P}\text{-a.s.}.$$

  As a corollary, taking the reciprocals, we obtain

$$\frac{m^2}{\Phi(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \frac{1}{\lambda} \qquad \mathbb{P}\text{-a.s.}, \tag{7.5}$$

  which holds in our setting since $\lambda > \lambda_c$.

- $f(z,\omega) = z$. Substituting in (7.4), we see that the sum on the LHS counts the number of nodes which have mark 1 in Γ_m. Indeed, we obtain

$$\frac{\Phi^+(\Gamma_m)}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda p \qquad \mathbb{P}\text{-a.s.}. \tag{7.6}$$

- Let C be the unique infinite cluster in $\mathcal{G}$. Using the ergodic theorem in (7.3) with $f(z,\omega) = \mathbb{1}\{\mathbf{0} \in C\}$, we see that the sum on the LHS counts the number of vertices of Φ which are present in the infinite cluster. Then, we have that

$$\frac{|C \cap \Gamma_m|}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda \, \theta(\lambda) \qquad \mathbb{P}\text{-a.s.}. \tag{7.7}$$

  Using the dominated convergence theorem (DCT) and (7.5), we also have that

$$\mathbb{E}\left[\frac{|C \cap \Gamma_m|}{\Phi(\Gamma_m)}\right] \overset{m\to\infty}{\longrightarrow} \theta(\lambda). \tag{7.8}$$

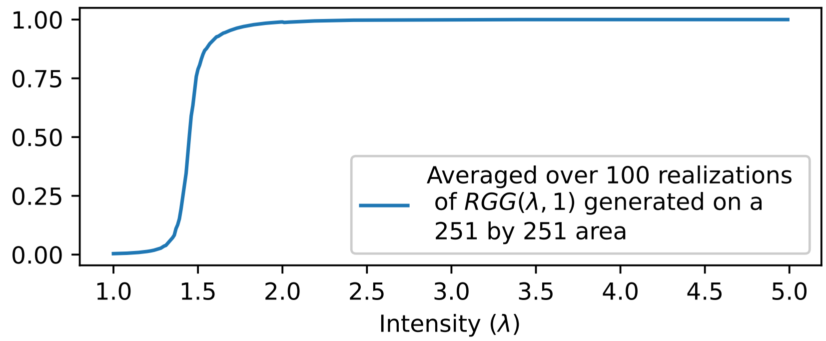


Figure 7.1: Percolation probability $\theta(\lambda)$ vs. intensity λ

This means that, for large m, the expected fraction of vertices of the infinite cluster within Γ_m is a good approximation for the percolation probability. We use this to obtain an empirical estimate of the percolation probability as follows. We generate 100 instantiations of the $RGG(\lambda, 1)$ model on Γ_{251}, for each value of λ between 1 and 5 (in steps of 0.01). The average number of vertices in the largest cluster within Γ_m normalized by the number of vertices within Γ_m is computed and taken as a proxy for the fraction of nodes within Γ_m of the infinite cluster. The graph obtained is shown in Fig. 7.1. We use the values from this plot in our numerical results.

- Suppose $\lambda p > \lambda_c$, so that $\mathcal{G}^+$ operates in the super-critical region. Let C^+ be the unique infinite cluster in $\mathcal{G}^+$. Since Φ^+ is a thinned point process of intensity λp, we can use the result from (7.7) for the infinite cluster C^+ to obtain

$$\frac{|C^+ \cap \Gamma_m|}{\nu(\Gamma_m)} \quad \xrightarrow{m\to\infty} \quad \lambda p\ \theta(\lambda p) \qquad \mathbb{P}\text{-a.s.}. \tag{7.9}$$

- Suppose that $\lambda p > \lambda_c$ and let C^{ext} denote the extended cluster of C^+, i.e. $C^{\text{ext}} = C^+ \cup \partial C^+$. Note that since C^+ is infinite, C^{ext} is also infinite. Hence, we refer to it as the *infinite extended cluster*, or IEC for short. Take $f(\omega) = \mathbb{1}(B_1(\mathbf{0}) \cap C(\Phi^+) \neq \emptyset)$. Observe that $\{X_i \in C^{\text{ext}}\} = \mathbb{1}(B_1(X_i) \cap C(\Phi^+) \neq \emptyset) = f(\theta_{X_i} w)$. So, using (7.3), we have that

$$\frac{1}{\nu(\Gamma_m)} \sum_{X_i \in \Gamma_m} \mathbb{1}\{X_i \in C^{\text{ext}}\} \overset{m\to\infty}{\longrightarrow} \lambda \mathbb{P}(B_1(\mathbf{0}) \cap C(\Phi^+) \neq \emptyset) \qquad \mathbb{P}\text{-a.s..}$$

By definition, $\mathbb{P}(B_1(\mathbf{0}) \cap C(\Phi^+) \neq \emptyset) = \theta(\lambda p)$, the percolation probability of Φ^+. We then have,

$$\frac{|C^{\text{ext}} \cap \Gamma_m|}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda\theta(\lambda p) \qquad \mathbb{P}\text{-a.s.} \tag{7.10}$$

Thus, it is natural to define, $\theta^{\text{ext}}(\lambda, p) := \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C^{\text{ext}}) = \theta(\lambda p)$.

Comparing RHS of (7.10) and (7.6) suggests an alternate viewpoint for the nodes that are present in the IEC. On the underlying point process Φ, define new iid marks $Z' \in \mathbb{K} = \{0, 1\}$ with $\text{Ber}(\theta^{\text{ext}}(\lambda, p))$ distribution. This means that a vertex is attributed mark 1, if it is in the IEC when probabilistic forwarding is carried out with forwarding probability p. Then, the fraction of nodes in the IEC when marks are Z corresponds to the fraction of nodes with mark 1 when marks are Z'. This interpretation will be useful in proposing a heuristic approach for probabilistic forwarding of multiple packets in Section 7.5.

7.3.3 Probabilistic forwarding of multiple packets

Consider now the probabilistic forwarding mechanism on n packets. Each node transmits a newly received packet with probability p independently of other packets. It is required to find the fraction of successful receivers, the nodes that receive at least k out of the n packets. From our discussion of probabilistic forwarding of a single packet (in Section 7.3.1), for large m, the number of nodes within Γ_m that receive a packet from the origin is well-approximated by the number of nodes in the IEC. In a similar way, the fraction of successful receivers within Γ_m can be well approximated by the fraction of nodes within Γ_m that are present in at least k out of the n IECs when probabilistic forwarding is carried out on the RGG, $\mathcal{G}^{\mathbf{0}}$. In this subsection, we will use the ergodic theorem and obtain explicit bounds on this fraction.

Equip each vertex of the point process Φ with mark $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_n) \in \mathbb{K} = \{0,1\}^n$. Here the j-th co-ordinate of the mark represents transmission of the j-th packet on Φ. More precisely, $Z_j(\cdot) \sim Ber(p)$ and, for two different vertices u and v, $\mathbf{Z}(X_u)$ and $\mathbf{Z}(X_v)$ are independent conditional on Φ. Therefore, it forms an iid marked point process. Define $C^{\text{ext}}_{k,n}$ to be the set of nodes which are present in at least k out of the n IECs. Taking $f(z,\omega) = \mathbf{1}\{\mathbf{0} \in C^{\text{ext}}_{k,n}\}$ in the statement of the ergodic theorem, we obtain

$$\frac{1}{\nu(\Gamma_m)} \sum_{X_i \in \Gamma_m} \mathbb{1}\{X_i \in C^{\text{ext}}_{k,n}\} \overset{m\to\infty}{\longrightarrow} \lambda\, \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C^{\text{ext}}_{k,n}) \qquad \mathbb{P}\text{-a.s..}$$

Denote by $\theta^{\text{ext}}_{k,n}(\lambda, p) := \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C^{\text{ext}}_{k,n})$. Then the above statement reads as

$$\lim_{m\to\infty} \frac{|C^{\text{ext}}_{k,n} \cap \Gamma_m|}{\nu(\Gamma_m)} = \lambda\, \theta^{\text{ext}}_{k,n}(\lambda, p) \qquad \mathbb{P}\text{-a.s..} \tag{7.11}$$

The results obtained using the ergodic theorems in (7.3) and (7.4) in this section are collected in Table 7.1 for quick reference.

7.4 Main results

In this section, we will obtain expressions for the expected fraction of successful receivers and the expected total number of transmissions on the finite graph G based on the framework that has been developed in the previous section.

While constructing $\mathcal{G}^{\mathbf{0}}$ (as described in Section 7.3.1), the graph corresponding to $\Gamma_{0,0}$ can be taken to be $G^{\mathbf{0}}_m$ (with additional edges from vertices in $\Gamma_{0,0}$ to those outside it). Alternately, $G^{\mathbf{0}}_m$ can be constructed by considering a restriction of $\mathcal{G} \sim RGG(\lambda, 1)$ to Γ_m and connecting the origin to nodes within $B_1(\mathbf{0})$. In essence, it is true that the distribution of nodes of $G^{\mathbf{0}}_m$ and $\mathcal{G}^{\mathbf{0}} \cap \Gamma_m$ is the same. Recall that the graph G on which the probabilistic forwarding mechanism is carried out, is the component of the origin in $G^{\mathbf{0}}_m$. In light of the correspondence between the vertices of $G^{\mathbf{0}}_m$ and $\mathcal{G}^{\mathbf{0}} \cap \Gamma_m$, the graph G should correspond to the graph induced on the nodes within Γ_m, which are present in the

Eq.	$f(z,\omega)$	Result		Using
(7.5)	1	$\dfrac{\Phi(\Gamma_m)}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda$	$\mathbb{P}$-a.s..	(7.3)
(7.6)	z	$\dfrac{\Phi^+(\Gamma_m)}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda p$	$\mathbb{P}$-a.s..	(7.4)
(7.7)	$\mathbf{1}\{\mathbf{0} \in C\}$	$\dfrac{\lvert C \cap \Gamma_m\rvert}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda\ \theta(\lambda)$	$\mathbb{P}$-a.s..	(7.3)
(7.9)	$\mathbf{1}\{\mathbf{0} \in C^+\}$	$\dfrac{\lvert C^+ \cap \Gamma_m\rvert}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda p\ \theta(\lambda p)$	$\mathbb{P}$-a.s..	(7.3)
(7.10)	$\mathbf{1}\{\mathbf{0} \in C^{\text{ext}}\}$	$\dfrac{\lvert C^{\text{ext}} \cap \Gamma_m\rvert}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda\theta(\lambda p)$	$\mathbb{P}$-a.s..	(7.4)
(7.11)	$\mathbf{1}\{\mathbf{0} \in C^{\text{ext}}_{k,n}\}$	$\dfrac{\lvert C^{\text{ext}}_{k,n} \cap \Gamma_m\rvert}{\nu(\Gamma_m)} \overset{m\to\infty}{\longrightarrow} \lambda\ \theta^{\text{ext}}_{k,n}(\lambda,p)$	$\mathbb{P}$-a.s..	(7.4)

Table 7.1: Results from evaluating the ergodic statements in (7.3) and (7.4) for different functions f.

cluster of the origin in $\mathcal{G}^{\mathbf{0}}$. However, these nodes also include those which are contained in the cluster of the origin through paths which go outside Γ_m, but are not connected to the origin within Γ_m. We refer to these as nodes in the cluster of the origin but *without a Γ_m-conduit* and denote them by $\widehat{C}_{\mathbf{0},m}$. We make the following assumption about $\widehat{C}_{\mathbf{0},m}$.

Assumption 2. $\displaystyle\lim_{m\to\infty} \frac{|\widehat{C}_{\mathbf{0},m}|}{m^2} = 0 \qquad \mathbb{P}\text{-}a.s.$

The assumption can be proved to be true over a subsequence since Markov inequality accompanied by the convergence in mean (similar to 6.4.1[1]), $\mathbb{E}\left[\frac{|\widehat{C}_{\mathbf{0},m}|}{m^2}\right] \overset{m\to\infty}{\longrightarrow} 0$, gives convergence in probability. However, a complete proof evades us.

Continuing, since $C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m = C_{\mathbf{0}}(G^{\mathbf{0}}_m) \cup \widehat{C}_{\mathbf{0},m}$ and $C_{\mathbf{0}}(G^{\mathbf{0}}_m) \cap \widehat{C}_{\mathbf{0},m} = \emptyset$, from the assumption, we obtain the following lemma.

[1]As in the case of a grid, the probability of there being a connected path in an annulus around Γ_m, is known to tend to 1 as $m \to \infty$ in the super-critical region, $\lambda > \lambda_c$.

Lemma 7.4.1. *For $\lambda > \lambda_c$, we have*

$$\lim_{m\to\infty} \frac{|C_{\mathbf{0}}(G^{\mathbf{0}}_m)|}{\lambda m^2} = \lim_{m\to\infty} \frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} \qquad \mathbb{P}\text{-}a.s.,$$

where $C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}})$ is the set of nodes in the cluster of the origin in $\mathcal{G}^{\mathbf{0}}$.

To get a handle on the fraction of nodes within Γ_m of $C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}})$, we will need the following lemma.

Lemma 7.4.2. *Let $A = \{\mathbf{0} \in C(\mathcal{G}^{\mathbf{0}})\}$, where $C(\mathcal{G}^{\mathbf{0}})$ is the infinite cluster of $\mathcal{G}^{\mathbf{0}}$. For $\lambda > \lambda_c$, we then have*

$$\lim_{m\to\infty} \frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} = \theta(\lambda)\mathbf{1}_A \qquad \mathbb{P}\text{-}a.s..$$

Proof. We can write

$$\frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} = \frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_A + \frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_{A^c}.$$

Since A^c is the event that the origin is in some finite cluster, the number of nodes within $C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}})$ is finite. In the limit as $m \to \infty$, the latter term on the RHS above goes to 0. For the first term, notice that $A = \{C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) = C(\mathcal{G}^{\mathbf{0}})\}$. This gives

$$\frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_A = \frac{|C(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_A.$$

Further, from (B.1),, we have that

$$\lim_{m\to\infty} \frac{|C(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} = \lim_{m\to\infty} \frac{|C(\mathcal{G}) \cap \Gamma_m|}{\lambda m^2} \qquad \mathbb{P}\text{-a.s..}$$

Therefore, using (7.7) in the RHS of the above equation, we obtain that

$$\begin{aligned}\lim_{m\to\infty} \frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} \mathbf{1}_A &= \lim_{m\to\infty} \frac{|C(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2} \mathbf{1}_A \\ &= \theta(\lambda) \mathbf{1}_A \qquad \mathbb{P}\text{-a.s..}\end{aligned}$$

□

Note: It should be noted here that the statements in Assumption 2 and Lemmas 7.4.1 and 7.4.2 hold $\mathbb{P}^{\mathbf{0}}$-a.s., since these are $\mathbb{P}$-a.s. statements made on the underlying graph $\mathcal{G}^{\mathbf{0}}$.

Before we proceed, we recall the definition of the minimum forwarding probability in (7.1):

$$p_{k,n,\delta} = \inf \left\{ p \;\middle|\; \mathbb{E}\left[\frac{R_{k,n}(G_m^{\mathbf{0}})}{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|} \right] \geq 1 - \delta \right\},$$

where the expectation is over the graph G and the probabilistic forwarding mechanism. Note that in our setting, the source, $\mathbf{0}$, always has mark 1 since it transmits all the n packets. To be more explicit, define $\mathbf{1} = (1, 1, \cdots, 1)$ to be the vector of all 1s of length n. We denote by $\mathbb{E}^{(\mathbf{0},\mathbf{1})}$, the expectation with respect to the Palm probability $\mathbb{P}^{\mathbf{0}}$ given a point at the origin, conditional on it having mark $\mathbf{Z}(\mathbf{0}) = \mathbf{1}$. In terms of this, the above equation then translates to

$$p_{k,n,\delta} = \inf \left\{ p \;\middle|\; \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|} \right] \geq 1 - \delta \right\}. \tag{7.12}$$

Next, since we are addressing a broadcast problem, it is necessary that a large fraction of nodes receive a packet. This, in turn necessitates that the fraction of nodes that transmit the packet is also large. With reference to the RGG on the whole plane, this means that the nodes in $\mathcal{G}^+$ need to have an infinite cluster. To allow for this, we make the following assumption.

Assumption 3. *The forwarding probability p is such that $\lambda p > \lambda_c$.*

Notice that the $p_{k,n,\delta}$ values obtained from simulations in Fig. 3.2 conform to this assumption. The assumption is discussed in slightly more detail in Section 7.6.2. We now obtain expressions for the minimum forwarding probability and the expected total number of transmissions based on these two assumptions.

7.4.1 Transmissions

Consider first the transmission of a single packet. Let $T(G_m)$ be the number of nodes of G_m that receive the packet from the source and transmit it and let $\mathcal{T}(\mathcal{G}) \cap \Gamma_m$ be the set of nodes within Γ_m that receive the packet from the source and transmit it when probabilistic forwarding is carried out on $\mathcal{G}$ [2]. From our construction, it follows that $T(G_m)$ is stochastically dominated by $|\mathcal{T}(\mathcal{G}) \cap \Gamma_m|$ since there might be nodes which receive a packet from outside Γ_m and transmit it. However, it can be shown that,

$$\lim_{m\to\infty} \frac{\mathbb{E}^{(\mathbf{0},1)}\left[T(G_m)\right]}{m^2} = \lim_{m\to\infty} \frac{\mathbb{E}^{(\mathbf{0},1)}\left[|\mathcal{T}(\mathcal{G}) \cap \Gamma_m|\right]}{m^2}.$$

This is because the expected fraction of transmitting nodes with no Γ_m-conduits diminishes as $m \to \infty$. Thus, it suffices to evaluate $\lim_{m\to\infty} \frac{\mathbb{E}^{(\mathbf{0},1)}[|\mathcal{T}(\mathcal{G})\cap\Gamma_m|]}{m^2}$ to find the expected number of transmissions for a single packet.

In the jargon of marked point processes, $\mathcal{T}(\mathcal{G})$ is the set of vertices with mark $Z(\cdot) = 1$ that are in the cluster containing the origin. Note that the origin has mark 1, since it always transmits the packet. As the vertices with mark 1 form a thinned point process, Φ^+ of intensity λp, $\mathcal{T}(\mathcal{G})$ is the set of nodes in the cluster containing the origin in $\mathcal{G}^+$. In Section 7.3.1, we denoted this set by $C^+_{\mathbf{0}}$. From Assumption 3, the graph on Φ^+ is in the super-critical regime and thus possesses a unique infinite cluster, C^+. The following theorem provides the expected size of $C^+_{\mathbf{0}} \cap \Gamma_m$. The proof proceeds by relating it to the expected size of $C^+ \cap \Gamma_m$ and using the ergodic result in (7.9). The detailed proof is provided in Section 7.7.1.

[2]It is implicit from the use of Palm probabilities that the origin is the source and probabilistic forwarding is formulated as an MPP as described in Section 7.3.1.

Theorem 7.4.3. *For $\lambda p > \lambda_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},1)} \left[\frac{|C_{\mathbf{0}}^{+} \cap \Gamma_m|}{\lambda m^2} \right] = p\,\theta(\lambda p)^2.$$

Therefore, for large values of m, the expected number of transmissions, $\mathbb{E}^0\left[T(G)\right]$, can be approximated by

$$\mathbb{E}^{(\mathbf{0},1)}\left[|C_{\mathbf{0}}^{+} \cap \Gamma_m|\right] \approx m^2 \lambda p\ \theta(\lambda p)^2.$$

Consider now the transmission of multiple packets. The n coded packets are transmitted independently of each other. The expected total number of transmissions of all n packets would just be n times the expected transmissions of a single packet. Therefore, from Theorem 7.4.3, we then obtain

$$\tau_{k,n,\delta} \approx n m^2 \lambda p_{k,n,\delta}\ \left(\theta(\lambda p_{k,n,\delta})\right)^2. \tag{7.13}$$

7.4.2 Minimum forwarding probability

In this section, we will obtain an expression for the minimum forwarding probability. Recall that this entails estimating $\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right]$, where $C_{\mathbf{0}}(G_m)$ is the set of nodes in the component of the origin in the underlying RGG on Γ_m and $R_{k,n}(G_m)$ are the number of nodes that receive at least k out of the n packets from the origin, which is the source. With reference to the discussion prior to Assumption 2, $C_{\mathbf{0}}(G_m)$ can be viewed as the set of nodes in the component of the origin in $\mathcal{G}^{\mathbf{0}}$ restricted to Γ_m but with only those nodes which are connected to the origin via Γ_m-conduits. $R_{k,n}(G_m)$ is the number of nodes among those in $C_{\mathbf{0}}(G_m)$, which are successful receivers. These arguments lets us think of the expectation $\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right]$, with respect to the RGG, $\mathcal{G}^{\mathbf{0}}$, instead of the finite RGG, $G_m^{\mathbf{0}}$.

Since we are interested in large dense networks, it is natural to assume that the origin is part of the infinite cluster of $\mathcal{G}^{\mathbf{0}}$. This means that the cluster of the origin in $G_m^{\mathbf{0}}$ connects to the infinite cluster in $\mathcal{G}^{\mathbf{0}}$ when $G_m^{\mathbf{0}}$ is embedded within it. In other words, the event $A = \{\mathbf{0} \in C(\mathcal{G}^{\mathbf{0}})\}$ occurs. The results of this section are made with this assumption, which is stated below explicitly. Additional justification for this is provided in Section 7.6.2.

Assumption 4. *The origin is part of the infinite cluster of* $\mathcal{G}^{\mathbf{0}}$.

From the discussion above and the assumption, our interest now is to estimate

$$\mathbb{E}_A^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right].$$

The subscript A in the expectation $\mathbb{E}_A^{(\mathbf{0},\mathbf{1})}$ indicates conditional expectation given that the event A occurs. From Assumption 3, it is clear that such a conditioning can indeed be done, since $\mathbb{P}(A) = \theta(\lambda) > 0$.

The following theorem gives the expected value of the fraction of successful receivers in the limit as $m \to \infty$ given the event A. Before we state the theorem, recall the formulation of probabilistic forwarding as a marked point process in Section 7.3. $C_{k,n}^{\text{ext}}$ was defined as the set of nodes which are present in at least k out of the n IECs and let $\theta_{k,n}^{\text{ext}} \equiv \theta_{k,n}^{\text{ext}}(\lambda, p) = \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C_{k,n}^{\text{ext}})$. Additionally, define $A_{[t]}^{\text{ext}}$ to be the event that the origin is present only in the IECs corresponding to the packets $1, 2, \cdots, t$.

Theorem 7.4.4. *For $\lambda p > \lambda_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}_A^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right] = \frac{1}{\theta(\lambda)^2}\sum_{t=k}^{n}\binom{n}{t}\theta_{k,t}^{ext}\ \mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_{[t]}^{ext}).$$

The proof is on similar lines as that on the grid in 6. It relies on carefully relating the fraction of successful receivers on G to the fraction of nodes present in at least k out of the n IECs corresponding to probabilistic forwarding on $\mathcal{G}^{\mathbf{0}}$. An outline of the proof is given in Section 7.7.2.

The following proposition is used to express $\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_{[t]}^{\text{ext}})$ in terms of $\theta_{k,n}^{\text{ext}}$.

Proposition 7.4.5.

$$\mathbb{P}^{(\mathbf{0},\mathbf{1})}\left(A^{ext}_{[t]}\right) = \begin{cases} \dfrac{\theta^{ext}_{t,n} - \theta^{ext}_{t+1,n}}{\binom{n}{t}} & 0 \leq t \leq n-1 \\ \theta^{ext}_{n,n} & t = n \end{cases}. \tag{7.14}$$

Proof. The second part follows directly from the definitions of $\theta^{\text{ext}}_{n,n}$ and the event $A^{\text{ext}}_{[n]}$. For the first part, define for $T \subseteq [n]$, A^{ext}_T to be the event that the origin is present in exactly the IECs indexed by T. Note that

$$\theta^{\text{ext}}_{k,n} = \mathbb{P}^{(\mathbf{0},\mathbf{1})}(\mathbf{0} \in C^{\text{ext}}_{k,n}) = \sum_{j=k}^{n} \sum_{\substack{T\subseteq[n] \\ |T|=j}} \mathbb{P}^{(\mathbf{0},\mathbf{1})}(A^{\text{ext}}_T).$$

Since the event A^{ext}_T depends only on the cardinality j (see Step 7 in Section 7.7.2), we obtain

$$\theta^{\text{ext}}_{k,n} = \sum_{j=k}^{n} \binom{n}{j} \mathbb{P}^{(\mathbf{0},\mathbf{1})}(A^{\text{ext}}_{[j]}).$$

We then have that $\theta^{\text{ext}}_{t,n} - \theta^{\text{ext}}_{t+1,n} = \binom{n}{t}\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A^{\text{ext}}_{[t]})$ for $0 \leq t \leq n-1$, which is the statement of the proposition. $\square$

We remark here that the statement of Theorem 7.4.4 can be used to obtain an estimate for the expected fraction of successful receivers without the conditioning on the event A. We write

$$\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right] = \theta(\lambda)\, \mathbb{E}^{(\mathbf{0},\mathbf{1})}_{A}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right] + (1-\theta(\lambda))\, \mathbb{E}^{(\mathbf{0},\mathbf{1})}_{A^C}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right]$$

Notice from Fig. 7.1 that $\theta(\lambda)$ shows a phase transition phenomenon. For the intensities we are interested in, $\mathbb{P}(A^c) = 1 - \theta(\lambda)$ is very small and the latter term in the above equation can be neglected. This also suggests that Assumption 4 is not a very strong requirement.

Consequently, for large m, using Theorem 7.4.4 and Proposition 7.4.5 in (7.12) yields

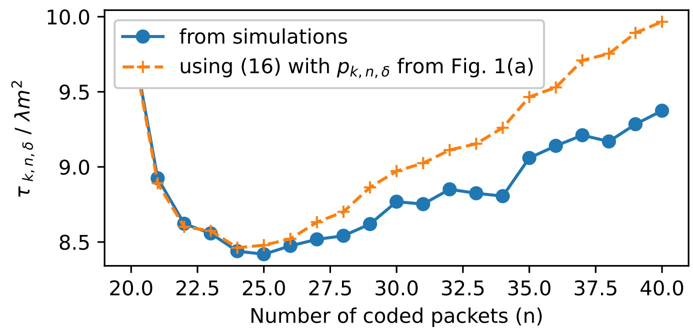


Figure 7.2: Comparison of the expected number of transmissions per node in the RGG(4.5, 1) model on Γ_{101} obtained using (7.13) with that obtained through simulations. Note that the $p_{k,n,\delta}$ value for each point on both the curves are from the simulations in Fig. 1(a).

an approximation for the minimum forwarding probability given by,

$$p_{k,n,\delta} \approx \inf \left\{ p \,\middle|\, \sum_{t=k}^{n-1} \frac{\theta^{\text{ext}}_{k,t}(\theta^{\text{ext}}_{t,n} - \theta^{\text{ext}}_{t+1,n})}{\theta(\lambda)} + \frac{\theta^{\text{ext}}_{k,n}\theta^{\text{ext}}_{n,n}}{\theta(\lambda)} \geq 1 - \delta \right\}. \tag{7.15}$$

7.4.3 Comparison with simulations

We have not been able to obtain exact expressions for the probability $\theta^{\text{ext}}_{k,t}(\lambda, p)$ in terms of the percolation probability $\theta(\lambda)$. However, in Section 7.6.1, we provide some bounds for it. We also develop an alternate heuristic approach, which provides comparable results for the minimum forwarding probability obtained through simulations, in Section 7.5.

Nevertheless, the approximation for the expected total number of transmissions, $\tau_{k,n,\delta}$ in (7.13) can be evaluated with the knowledge of the minimum forwarding probability. In Fig. 7.2, we show the plot of $\tau_{k,n,\delta}$ (normalized by λm^2) with n in which we use $p_{k,n,\delta}$ values from Fig. 3.2(a)

It is observed that for $n \lesssim 26$, both the curves match pretty well. However, for $n > 26$ they diverge. This can be attributed to the fact that as n increases, $p_{k,n,\delta}$ decreases as in Fig 3.2(a) and thus $\lambda p_{k,n,\delta} \searrow \lambda_c$. The estimate for the percolation probability, $\theta(\lambda)$, obtained via the ergodic result in (7.8) may not be accurate near the critical intensity, λ_c (which is itself not exactly known). In particular, Γ_{251} may not be large enough for the ergodic result in (7.8) to kick in, as we approach λ_c.

Nevertheless, this provides justification to our observation that the expected number of transmissions indeed decreases when we introduce coded packets along with probabilistic forwarding. This comes with a catch that the minimum forwarding probability for a near-broadcast behaves as in Fig 3.2(a). In order to establish this, we provide a heuristic explanation for it in the next section.

7.5 A heuristic argument

In the marked point process formulation, probabilistic forwarding of multiple packets was modeled using marks given by $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_n)$ with $Z_i \sim Ber(p)$ on the underlying point process Φ. We refer to this as the *original model*. Motivated by the alternate interpretation for the nodes in the IEC expounded at the end of Section 7.3.2, in this section, we provide a heuristic approach for evaluating the minimum forwarding probability.

As before, let $\theta^{\text{ext}}(\lambda, p)$ denote the probability that the origin is in the IEC for a single packet transmission. Associate a new mark $\mathbf{Z}' = (Z_1', Z_2', \cdots, Z_n') \in \mathbb{K} = \{0,1\}^n$ to each vertex of Φ. The $i-$th co-ordinate of $\mathbf{Z}'$ corresponds to probabilistic forwarding of the $i-$th packet. The mark $\mathbf{Z}'$ is chosen such that each of the i co-ordinates is either 1 with probability $\theta^{\text{ext}}(\lambda, p)$ $(= \theta(\lambda p))$ or 0 with the remaining probability, independent of the others. Similar to the viewpoint for the single packet transmission, our idea is to use Z_i' as a proxy for a vertex to be present in the IEC in probabilistic forwarding of the $i-$th packet. We refer to this as the *mean-field model.*

There are two key differences between the two models defined here. Firstly, in the original model, presence of a node in the IEC is not independent of other nodes being present in the IEC. Whereas, in the mean-field model, $Z_i'(\mathbf{u})$ and $Z_i'(\mathbf{v})$ are chosen to be independent $Ber(\theta(\lambda p))$ random variables for two distinct vertices $\mathbf{u}$ and $\mathbf{v}$. Since Z_i' is interpreted as an indicator whether a vertex is present in the $i-$th IEC, this independence is enforced. Secondly, in the original model, presence of a particular node in IECs corresponding to two different packets, are not independent. They are independent conditional on Φ but not otherwise. In the mean-field model, since $\mathbf{Z}_i'(\mathbf{v})$ and $\mathbf{Z}_j'(\mathbf{v})$ are taken to be iid, this dependence is over-looked.

To analyze the mean-field model, let us use Theorem 7.2 with

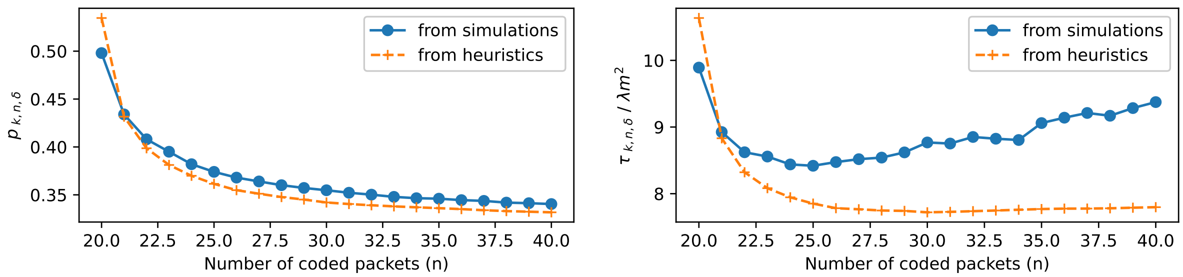
$$f(\mathbf{z}',\omega)=\sum_{j=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=j}}\prod_{i\in T}z'_i\prod_{i\notin T}(1-z'_i).$$

The inner summation is 1 only if a node has mark 1 in exactly the co-ordinates indexed by T (which has cardinality j). Since the outer sum goes over all $j\geq k$, the value of the function is 1 for a vertex which has mark 1, in at least k out of the n co-ordinates. From our interpretation of $\mathbf{Z}'$, the value of the function, f, for a vertex is equal to 1 if it is present in at least k out of the n IECs of the original model. Define $C'_{k,n}$ to be the set of nodes which have mark $Z'_i(\cdot)=1$ in at least k out of the n packet transmissions in the mean-field model. Here, $C'_{k,n}$ acts as a proxy for $C^{\text{ext}}_{k,n}$. Since $f(\mathbf{Z}'(\mathbf{v}),\omega)=1$ if $\mathbf{v}\in C'_{k,n}$, we can apply Theorem 7.2, to obtain for $\mathbb{P}$ almost surely

$$\frac{1}{\nu(\Gamma_m)}\sum_{X_i\in\Gamma_m}\mathbb{1}\{X_i\in C'_{k,n}\}\overset{m\to\infty}{\longrightarrow}\lambda\sum_{\mathbf{z}'\in\{0,1\}^n}\mathbb{P}(\mathbf{Z}'=\mathbf{z}')\;\mathbb{E}^{(\mathbf{0},\mathbf{z}')}\left[\sum_{j=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=j}}\prod_{i\in T}\mathbf{Z}'_i\prod_{i\notin T}(1-\mathbf{Z}'_i)\right]$$
$$=\lambda\sum_{j=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=j}}\sum_{\mathbf{z}\in\{0,1\}^n}\mathbb{P}(\mathbf{Z}'=\mathbf{z}')\times\prod_{i\in T}\mathbf{z}'_i\prod_{i\notin T}(1-\mathbf{z}'_i).$$

For a fixed j and a set T with $|T|=j$, there is exactly one $\mathbf{z}'$ such that $\prod_{i\in T}\mathbf{z}'_i\prod_{i\notin T}(1-\mathbf{z}'_i)=1$ and the probability of such a $\mathbf{z}'$ is given by $\mathbb{P}(\mathbf{Z}'=\mathbf{z}')=\theta^{\text{ext}}(\lambda,p)^j\times(1-\theta^{\text{ext}}(\lambda,p))^{n-j}$. Thus, the expression above reduces to

$$\frac{|C'_{k,n}\cap\Gamma_m|}{\nu(\Gamma_m)}\overset{m\to\infty}{\longrightarrow}\lambda\sum_{j=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=j}}\theta^{\text{ext}}(\lambda,p)^j(1-\theta^{\text{ext}}(\lambda,p))^{n-j}$$
$$=\lambda\sum_{j=k}^{n}\binom{n}{j}\theta^{\text{ext}}(\lambda,p)^j(1-\theta^{\text{ext}}(\lambda,p))^{n-j}\qquad\mathbb{P}\text{-a.s.}$$



(a) Minimum retransmission probability (b) Expected total number of transmissions

Figure 7.3: Comparison of simulation results with results obtained in (7.16) and (7.13) on $RGG(4.5, 1)$ on Γ_{101} with $k = 20$ packets and $\delta = 0.1$

Define

$$\theta'_{k,n} \equiv \theta'_{k,n}(\lambda, p) = \sum_{j=k}^{n} \binom{n}{j} \theta(\lambda p)^j (1 - \theta(\lambda p))^{n-j}.$$

From our interpretation of $C'_{k,n}$ as representing $C^{\text{ext}}_{k,n}$ of the original model, we use $\theta'_{k,n}$ instead of $\theta^{\text{ext}}_{k,n}$ in (7.15), and after a series of manipulations, the minimum forwarding probability obtained via this heuristic approach, $p'_{k,n,\delta}$, would be the minimum probability p such that

$$\frac{1}{\theta(\lambda)} \sum_{t=k}^{n} \sum_{j=k}^{t} \binom{n}{t} \binom{t}{j} \theta(\lambda p)^{t+j} (1 - \theta(\lambda p))^{n-j} \geq 1 - \delta.$$

This expression is similar to the expression that was obtained for the case of a grid in 6.3.2. Using Proposition 6.4.4, we then have

$$p'_{k,n,\delta} = \inf \left\{ p \;\middle|\; \frac{\mathbb{P}(Y \geq k)}{\theta(\lambda)} \geq 1 - \delta \right\} \tag{7.16}$$

where $Y \sim \text{Bin}(n, (\theta(\lambda p))^2)$.

The $p'_{k,n,\delta}$ values obtained using this expression is compared alongside the simulation results in Fig. 7.3(a). The expected total number of transmissions obtained via (7.13) is plotted in Fig. 7.3(b). The simulation setup is the same as described in Section 3 for the intensity $\lambda = 4.5$.

It is observed that the curve for the minimum forwarding probability obtained via our analysis tracks the simulation curve pretty well. However, the curve for the expected total number of transmissions deviates from the simulation results substantially for larger values of n. This can be attributed to the drastic change in $\theta(\lambda)$ around the critical intensity λ_c. Even though there seems to be a minor difference in the forwarding probability of the original and the mean-field model, the behaviour of the percolation probability around λ_c creates a huge divide between the two transmission plots in Fig 7.3(b) . This behaviour is similar to what was obtained on the grid in Section 6.3.2. Nevertheless, note that the $\tau_{k,n,\delta}$ curve initially decreases to a minimum and then gradually increases with n (albeit very slowly). This shows that probabilistic forwarding with coding is indeed beneficial on RGGs in terms of the number of transmissions required for a near-broadcast.

7.6 Discussion

7.6.1 Bounds on $\theta_{k,n}^{\text{ext}}$

We give two lower bounds for $\theta_{k,n}^{\text{ext}}(\lambda, p)$. The probability $\theta_{k,n}^{\text{ext}}(\lambda, p)$ can be expressed in terms of the events A_T^{ext} as follows.

$$\theta_{k,n}^{\text{ext}}(\lambda, p) = \mathbb{P}^{\mathbf{0}}\left(\bigcup_{|T|\geq k} A_T^{\text{ext}}\right) = \sum_{|T|\geq k} \mathbb{P}^{\mathbf{0}}(A_T^{\text{ext}})$$

A simple lower bound for $\theta_{k,n}^{\text{ext}}(\lambda, p)$ can be obtained by taking the term corresponding to $T = [n]$ in the above summation.

$$\begin{aligned}
\theta_{k,n}^{\text{ext}}(\lambda, p) \geq \mathbb{P}^{\mathbf{0}}(A_{[n]}^{\text{ext}}) &= \mathbb{P}^{\mathbf{0}}\left(\bigcap_{i=1}^{n}\{\mathbf{0} \in C_{\infty,i}^{\text{ext}}\}\right) \\
&\overset{(a)}{\geq} \prod_{i=1}^{n} \mathbb{P}^{\mathbf{0}}\left(\mathbf{0} \in C_{\infty,i}^{\text{ext}}\right) \\
&= \mathbb{P}^{\mathbf{0}}\left(\mathbf{0} \in C_{\infty,i}^{\text{ext}}\right)^n
\end{aligned}$$

Here, the inequality in (a) is via the FKG inequality since the events $\{\mathbf{0} \in C^{\text{ext}}_{\infty,i}\}$ are increasing events. This gives

$$\theta^{\text{ext}}_{k,n}(\lambda, p) \geq \theta(\lambda p)^n. \tag{7.17}$$

Notice that, this along with Assumption 3 suffices to ensure that our analysis yields non-trivial results for all values of k and n.

We now provide a second bound. For this, recall the iid marked point process Φ equipped with the mark structure $\mathbf{Z}$. Define a new marked point process Φ_T with the underlying point process Φ and marks $Z_T = \prod_{i \in T} Z_i \prod_{j \notin T}(1 - Z_j)$. The points with mark 1 in Φ_T, form a thinned version of Φ where each vertex is retained with probability $\mathbb{P}(Z_T = 1|\Phi) = \mathbb{P}(Z_i = \mathbf{1}\{i \in T\}\ ,\ \ i \in [n]|\Phi) = p^{|T|}(1-p)^{n-|T|}$. Thus Φ_T is an iid marked point process with $Ber(p^{|T|}(1-p)^{n-|T|})$ marks.

Let $C^{\text{ext}}(\Phi_T)$ denote the IEC of Φ_T. Notice that

$$\bigcup_{|T| \geq k} \{\mathbf{0} \in C^{\text{ext}}(\Phi_T)\} \subseteq \{\mathbf{0} \in C^{\text{ext}}_{k,n}\}.$$

The probability of the event in the LHS above can be found as

$$\begin{aligned} \mathbb{P}^{\mathbf{0}}\left(\bigcup_{|T| \geq k} \{\mathbf{0} \in C^{\text{ext}}(\Phi_T)\}\right) &= 1 - \mathbb{P}^{\mathbf{0}}\left(\bigcap_{|T| \geq k} \{\mathbf{0} \notin C^{\text{ext}}(\Phi_T)\}\right) \\ &= 1 - \prod_{j=k}^{n} \left(1 - \theta^{\text{ext}}(\lambda, p^j(1-p)^{n-j})\right)^{\binom{n}{j}} \end{aligned}$$

Therefore, the probability $\theta^{\text{ext}}_{k,n}(\lambda, p)$ can be bounded as

$$\theta^{\text{ext}}_{k,n}(\lambda, p) \geq 1 - \prod_{j=k}^{n} \left(1 - \theta(\lambda p^j(1-p)^{n-j})\right)^{\binom{n}{j}} \tag{7.18}$$

7.6.2 A note on our assumptions

In this subsection, we provide some justifications for the assumptions made in our analysis. Our interest in this thesis is to broadcast information on large and dense networks. A basic requirement for this is that a large number of nodes in the network must be reachable from the origin. In the sub- critical regime, i.e. $\lambda < \lambda_c \approx 1.44$, the clusters are finite and small. To model large dense ad-hoc networks, we need the graph to be connected on a large area Γ_m. This necessitates λ to be in the super-critical regime and the component of the origin within Γ_m to be large. In the limit as $m \to \infty$, this requires that the origin be present in the infinite cluster of the underlying RGG, thus justifying Assumption 4.

Further, notice that for a near-broadcast, we need the expected fraction of successful receivers to be close to 1, i.e., $\mathbb{E}^{\mathbf{0}}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G}^{\mathbf{0}})\cap\Gamma_m|}{\lambda\theta(\lambda)m^2}\right] \geq 1-\delta$ for some small $\delta > 0$ (The denominator here is the expected number of nodes within Γ_m of the infinite cluster C.). If we would like this to hold for sufficiently large m, then the forwarding probability must be such that $\mathcal{R}_{k,n}(\mathcal{G}^{\mathbf{0}})$ has infinite cardinality. This implies that p must be such that there is an IEC during probabilistic forwarding on $\mathcal{G}^{\mathbf{0}}$. Now, since existence of an IEC implies existence of an infinite cluster, the p value must ensure presence of an infinite cluster. Thus $\lambda p > \lambda_c$. This justifies Assumption 3.

It can also be seen from the simulation results in Fig. 3.2 that $\tau_{k,n,\delta}$ is minimized when the forwarding probability is such that $\lambda p_{k,n,\delta} > \lambda_c$ or $p_{k,n,\delta} > 0.32$. Further, results obtained from our heuristic approach in Fig. 7.3(a) and Fig. 7.3(b) also suggest that the expected total number of transmissions is indeed minimized when operating in the super-critical regime.

7.7 Proofs

In this section, we collect the proofs of Theorem 7.4.3 and Theorem 7.4.4.

7.7.1 Proof of Theorem 7.4.3

Theorem 7.4.3. *For $\lambda p > \lambda_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^{+}\cap\Gamma_m|}{\lambda m^2}\right] = p\,\theta(\lambda p)^2.$$

Proof. The proof is along the same lines as that of Lemma 7.4.2. Denote by C^{+}, the unique infinite cluster of the thinned process Φ^+. Define the event $A = \{\mathbf{0} \in C^+\} = \{B_1(\mathbf{0}) \cap C^+ \neq \emptyset\} \cap \{Z(\mathbf{0}) = 1\}$. We then have

$$\mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\right] = \mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}\right] + \mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A^c\}\right]. \tag{7.19}$$

As before, the latter term goes to 0 as $m \to \infty$. This is because A^c is the event that the cluster containing the origin $C_{\mathbf{0}}^+$ is finite. Using DCT, we obtain

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A^c\}\right] = 0 \tag{7.20}$$

For the first term on the RHS of (7.19), notice that $A = \{C_{\mathbf{0}}^+ = C^+\}$. This gives

$$\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\} = \frac{|C^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}. \tag{7.21}$$

$\frac{|C^+ \cap \Gamma_m|}{\Phi(\Gamma_m)}$ is the fraction of vertices of the infinite cluster of Φ^+ within Γ_m. Using (7.9), we have that

$$\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\} \overset{m\to\infty}{\longrightarrow} p\,\theta(\lambda p)\mathbb{1}\{A\} \qquad \mathbb{P}\text{-a.s.}$$

Using DCT, the above result extends to the expected value as well, i.e.,

$$\begin{aligned}\lim_{m\to\infty} \mathbb{E}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}\right] &= p\;\; \theta(\lambda p)\mathbb{P}(A)\\ &= p^2\,\theta(\lambda p)^2,\end{aligned}$$

where the last identity is because from the definition of A, $\mathbb{P}(A) = \mathbb{P}(B_1(\mathbf{0}) \cap C^+ \neq \emptyset)\mathbb{P}(Z(\mathbf{0}) = 1) = p\theta(\lambda p)$ and we have also used that $\{B_1(\mathbf{0}) \cap C^+ \neq \emptyset\}$ and $\{Z(\mathbf{0}) = 1\}$ are independent events with respect to $\mathbb{P}$. Moreover, from Proposition B.1.1 of Appendix B, the same holds for the expectation with respect to the Palm probability. Thus,

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{|C_{\mathbf{0}}^+ \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}\right] = p^2\,\theta(\lambda p)^2.$$

The proof is complete by noting that if $Z(\mathbf{0}) = 0$, then $C_{\mathbf{0}}^{+} = \emptyset$ and so

$$\mathbb{E}^{\mathbf{0}}\left[\frac{|C_{\mathbf{0}}^{+} \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}\right] = p\,\mathbb{E}^{(\mathbf{0},1)}\left[\frac{|C_{\mathbf{0}}^{+} \cap \Gamma_m|}{\lambda m^2}\mathbb{1}\{A\}\right].$$

□

7.7.2 Proof of Theorem 7.4.4

In this section, we provide a sketch of the proof for Theorem 7.4.4.

Theorem 7.4.4. *For $\lambda p > \lambda_c$, we have*

$$\lim_{m\to\infty} \mathbb{E}_A^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\right] = \frac{1}{\theta(\lambda)^2}\sum_{t=k}^{n}\binom{n}{t}\theta_{k,t}^{ext}\,\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_{[t]}^{ext}).$$

Sketch of proof: **Step 1:** We first evaluate

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{|C_{\mathbf{0}}(G_m)|}\mathbf{1}_A\right] = \lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)\mathbf{1}_A}{|C_{\mathbf{0}}(G_m)|\mathbf{1}_A}\right]$$

and then divide it by $\mathbb{P}(A) = \mathbb{P}(\mathbf{0} \in C(\mathcal{G}^0)) = \theta(\lambda)$ to obtain the required conditional expectation. We take the convention that $\frac{0}{0} = 0$. Note that Assumption 3 ensures that $\theta(\lambda) > 0$.

Step 2: Specializing the statement of Lemma 7.4.1 on the event A, we obtain

$$\lim_{m\to\infty}\frac{|C_{\mathbf{0}}(G_m^{\mathbf{0}})|}{\lambda m^2}\mathbf{1}_A = \lim_{m\to\infty}\frac{|C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_A \qquad \mathbb{P}\text{-a.s..}$$

Notice that on the event A, $C_{\mathbf{0}}(\mathcal{G}^{\mathbf{0}}) = C(\mathcal{G}^{\mathbf{0}})$. Using (B.1), (7.7) and the note following Lemma 7.4.2, we have for $\lambda > \lambda_c$

$$\begin{aligned}\lim_{m\to\infty}\frac{|C_{\mathbf{0}}(G_m)|}{\lambda m^2}\mathbf{1}_A &= \lim_{m\to\infty}\frac{|C(\mathcal{G}) \cap \Gamma_m|}{\lambda m^2}\mathbf{1}_A \\ &= \theta(\lambda)\mathbf{1}_A \qquad \mathbb{P}^{\mathbf{0}}\text{-a.s..}\end{aligned}$$

Conditional on the mark of the origin $\mathbf{Z}(\mathbf{0}) = \mathbf{1}$, we have

$$\lim_{m\to\infty} \frac{|C_{\mathbf{0}}(G_m)|}{\lambda m^2}\mathbf{1}_A = \theta(\lambda)\mathbf{1}_A \qquad \mathbb{P}^{(\mathbf{0},\mathbf{1})}\text{-a.s.}$$

Step 3: Let $\mathcal{R}_{k,n}(\mathcal{G})$ be the set of nodes that receive at least k out of the n packets from the origin when probabilistic forwarding is carried out on $\mathcal{G}$. Using arguments similar to those following Assumption 2 for nodes without Γ_m-conduits, we have that

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{\lambda m^2}\mathbf{1}_A\right] = \lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G})\cap\Gamma_m|}{\lambda m^2}\mathbf{1}_A\right]. \tag{7.22}$$

Step 4: For $T \subseteq [n]$, let A_T^{ext} be the event that the origin is present in exactly the IECs indexed by T. Conditioning on the event A_T^{ext}, we obtain

$$\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G})\cap\Gamma_m|}{\lambda m^2}\mathbf{1}_A\right] = \sum_{t=0}^{n}\sum_{\substack{T\subseteq[n]\\|T|=t}} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G})\cap\Gamma_m|}{\lambda m^2}\mathbf{1}_A\middle| A_T^{\text{ext}}\right]\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_T^{\text{ext}}). \tag{7.23}$$

If $|T| < k$, then the nodes of $\mathcal{R}_{k,n}(\mathcal{G})$ within Γ_m must reside in finite clusters whose fraction vanishes in the limit of large m. If $|T| \geq k$, then it is only the nodes which are within at least k IECs among those packet transmissions which are indexed by T, that contribute towards the expectation. Denote such nodes by $\mathcal{R}_{k,T}$. The remaining nodes of $\mathcal{R}_{k,n}(\mathcal{G})$ within Γ_m, must be in at least one finite cluster and hence their fraction vanishes in the limit. Additionally, given A_T^{ext} for $|T| > 0$, the $\mathbf{0}$ must be present in the infinite cluster of the underlying graph i.e., $\mathbf{1}_A = 1$. Putting all these together, we obtain

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G})\cap\Gamma_m|}{\lambda m^2}\mathbf{1}_A\right] = \lim_{m\to\infty}\sum_{t=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=t}} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle| A_T^{\text{ext}}\right]\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_T^{\text{ext}}). \tag{7.24}$$

Step 5: Define $\mathbf{O}$ to be the event that the origin has mark 1 in all the n packet transmissions. The expectation on the RHS in the above equation can be written as

$$\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{\text{ext}}\right]\mathbb{E}^{\mathbf{0}}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{\text{ext}}\cap\mathbf{O}\right].$$

$\mathcal{R}_{k,T}$ is independent of the packet transmissions which are not in T. The event $\mathbf{O}$ can be thus restricted to only those indices in T. However, the conditioning event $A_T^{\text{ext}}\cap\mathbf{O}$ is then the event that $\mathbf{0}$ is in the infinite cluster C^+ in the packet transmissions indexed by T. Call this event A_T^+. We then have

$$\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{\text{ext}}\right]=\mathbb{E}^{\mathbf{0}}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{+}\right]\tag{7.25}$$

Step 6: Conditional on the event A_T^+, the set $\mathcal{R}_{k,T}$ has the same distribution as the set $C_{k,|T|}^{\text{ext}}$, which was defined in Section 7.3.3. This gives

$$\mathbb{E}^{\mathbf{0}}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{+}\right]=\mathbb{E}^{\mathbf{0}}\left[\frac{|C_{k,|T|}^{\text{ext}}\cap\Gamma_m|}{\lambda m^2}\right].$$

From Proposition B.1.4 of Appendix B, by taking limits as $m\to\infty$, the expectation with respect to the Palm probability, $\mathbb{E}^{\mathbf{0}}$, can be written in terms of the expectation $\mathbb{E}$, yielding

$$\lim_{m\to\infty}\mathbb{E}^{\mathbf{0}}\left[\frac{|\mathcal{R}_{k,T}\cap\Gamma_m|}{\lambda m^2}\middle|A_T^{+}\right]=\lim_{m\to\infty}\mathbb{E}\left[\frac{|C_{k,|T|}^{\text{ext}}\cap\Gamma_m|}{\lambda m^2}\right]\tag{7.26}$$

Step 7: Using (7.11) with n replaced by $|T|=t$ and employing DCT, we obtain

$$\lim_{m\to\infty}\mathbb{E}\left[\frac{|C_{k,|T|}^{\text{ext}}\cap\Gamma_m|}{\lambda m^2}\right]=\theta_{k,t}^{\text{ext}}(\lambda,p)\tag{7.27}$$

Step 8: Clubbing the expressions from (7.25), (7.26) and (7.27) into (7.24), and using (7.22), we obtain

$$\lim_{m\to\infty}\mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{\lambda m^2}\mathbf{1}_A\right]=\sum_{t=k}^{n}\sum_{\substack{T\subseteq[n]\\|T|=t}}\theta_{k,t}^{\text{ext}}\,\mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_T^{\text{ext}}).$$

Step 9: The event A_T^{ext} can be expressed as

$$A_T^{\text{ext}} = \bigcap_{i \in T} \{\mathbf{0} \in C_i^{\text{ext}}\} \bigcap_{j \notin T} \{\mathbf{0} \notin C_j^{\text{ext}}\}.$$

Here, $C_1^{\text{ext}}, C_2^{\text{ext}}, \cdots, C_n^{\text{ext}}$ denote the IECs corresponding to the n packet transmissions. Since $\{\mathbf{0} \in C_i^{\text{ext}}\} = \{B_1(\mathbf{0}) \cap C_i^{+} \neq \emptyset\}$, the event A_T^{ext} does not depend on the specific mark of $\mathbf{0}$. Furthermore, the event A_T^{ext} does not depend on the specific choice of the set T, but just on the cardinality $|T|$. This is because a relabeling of the packets does not alter the probability of A_T^{ext}. For a particular value of $|T| = t$, define

$$A_{[t]}^{\text{ext}} = \bigcap_{i=1}^{t} \{\mathbf{0} \in C_i^{\text{ext}}\} \bigcap_{j=t+1}^{n} \{\mathbf{0} \notin C_j^{\text{ext}}\}.$$

Notice now that the terms within the summation in Step 7, $\theta_{k,t}^{\text{ext}}\ \mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_T^{\text{ext}})$ are identical for different T with the same cardinality. Therefore,

$$\lim_{m\to\infty} \mathbb{E}^{(\mathbf{0},\mathbf{1})}\left[\frac{R_{k,n}(G_m)}{\lambda m^2}\mathbf{1}_A\right] = \sum_{t=k}^{n} \binom{n}{t}\ \theta_{k,t}^{\text{ext}}\ \mathbb{P}^{(\mathbf{0},\mathbf{1})}(A_T^{\text{ext}}).$$

Step 10: Putting together the results from Step 2 and Step 9 and dividing by $\theta(\lambda)$ gives the statement of the theorem. □

Chapter 8

Preliminary Investigation on Random Regular Graphs

In this chapter, we discuss some aspects of the performance of the probabilistic forwarding mechanism on random d-regular graphs. Random regular graphs (RRGs) come under a broad class of random graphs which are specified by their degree sequences. For a graph, with vertex set $V_m = \{1, \cdots, m\}$ having m vertices, a degree sequence $\mathbf{d} = \mathbf{d}(m)$ specifies the degree of each of the m vertices. In particular, $\mathbf{d}(m) = (d_1, \cdots, d_m)$, where vertex i has degree d_i, and $\sum_{i=1}^{m} d_i$ is even. A random graph $\mathcal{G}_m(\mathbf{d})$, with degree sequence $\mathbf{d}$ is obtained by choosing a graph uniformly from all simple graphs with this degree sequence.

Probabilistic forwarding on $\mathcal{G}_m(\mathbf{d})$ is carried out by first choosing a source node uniformly at random from the m vertices. As in the case of random geometric graphs (RGGs), the broadcast mechanism is employed on the component of the source, i.e., the source transmits all n packets to its one-hop neighbours and every other node in the component of the source forwards a packet with probability p independently of other packets and other nodes. Denote the set of nodes in the component of the source by $C_0(\mathcal{G}_m(\mathbf{d}))$. It is now possible to define the minimum forwarding probability and the expected total number of transmissions as in the case of the RGG. Let $\mathcal{R}_{k,n}(\mathcal{G}_m(\mathbf{d}))$ be the successful receivers which receive at least k out of the n packets from the source. Then

$$p_{k,n,\delta} = \inf \left\{ p \;\middle|\; \mathbb{E}\left[\frac{|\mathcal{R}_{k,n}(\mathcal{G}_m(\mathbf{d}))|}{|C_0(\mathcal{G}_m(\mathbf{d}))|} \right] \geq 1 - \delta \right\}$$

and $\tau_{k,n,\delta}$ is the expected total number of transmissions when probabilistic forwarding is carried out with probability $p_{k,n,\delta}$. The expectation in both these definitions is with respect to the probabilistic forwarding mechanism, choice of the source and the randomness in the underlying graph.

We remark here that practical wireless networks are seldom deployed keeping the degree sequence in mind. Our interest in considering probabilistic forwarding on such networks is purely for theoretical reasons. Recall from our analysis on grids and RGGs, that the site percolation process is a faithful model of the probabilistic forwarding mechanism. On graphs with a specified degree sequence, the site percolation mechanism results in another uniformly generated graph from a degree sequence which depends on the probability with which each site is declared open. This makes such graphs amenable to analysis. Site percolation on graphs with a given degree sequence has been studied in [90–93]. In the following section, we put together results from these papers to characterize the probabilistic forwarding algorithm on graphs with a given degree sequence, in the regime where the number of coded packets n is large. We then specialize these results to random regular graphs.

It should be remarked here that we only consider the problem of quantifying the expected number of transmissions on the RRG here. Obtaining estimates for the minimum forwarding probability involves approximating the expected fraction of successful receivers, which is a harder problem. We leave this for future work.

8.1 Random graphs with a specified degree sequence

A popular way to construct graphs with a given degree sequence is via the configuration model which has been described for RRGs in Section 3.4.2. For a graph with a general degree sequence $\mathbf{d}$, create a set of points $\mathbf{P} = \{1 \times [d_1], 2 \times [d_2], \cdots, m \times [d_m]\}$, where

$[d_i] = \{1, 2, \cdots, d_i\}$. This set contains $\sum_{i=1}^{m} d_i$ points, each corresponding to a half-edge (or a stub) emanating from a vertex. Let M be a uniformly random perfect matching of the points in $\mathbf{P}$. We obtain the (multi)graph, $\mathcal{G}_m^*(\mathbf{d})$ by projecting $\mathbf{P}$ onto V_m preserving adjacencies, i.e., for any two vertices $i, j \in V_m$, if M contains an edge between a point in $i \times [d_i]$ and a point in $j \times [d_j]$, then $\mathcal{G}_m^*(\mathbf{d})$ contains the edge (i, j). This process might result in a graph with multiple edges and loops, but upon conditioning on it being a simple graph, we obtain a uniformly generated simple graph from the degree sequence, $\mathbf{d}$. Denote this graph by $\mathcal{G}_m(\mathbf{d})$.

The usual paradigm while discussing these graphs is to consider a sequence of graphs, $\{\mathcal{G}_m\}_{m\in\mathbb{Z}^+}$ as done in [90]. In particular $\mathcal{G}_m \equiv \mathcal{G}_m(\mathbf{d}^{(m)})$ for each $m \geq 1$, is a uniformly random graph on the set $V_m = \{1, \cdots, m\}$ having a degree sequence $\mathbf{d}^{(m)} = \mathbf{d}(m) = (d_1, \cdots, d_m)$, i.e., vertex i has degree d_i and $\sum_{i=1}^{m} d_i$ is even. Without loss of generality, we take $d_1 \leq \cdots \leq d_m$ and define $\Delta = \max_{1\leq i\leq m} d_i = d_m$. Further, let $D_i(m) = |\{j \in V_n : d_j = i\}|$ be the number of vertices with degree i. An *asymptotic degree sequence* is a sequence $(\mathbf{d}(m))_{m\in\mathbb{Z}^+}$ where for each $m \in \mathbb{Z}^+$ the vector $\mathbf{d}(m)$ is a degree sequence on V_m. An asymptotic degree sequence is *sparse* if for every $i \in \mathbb{N}$, we have $\lim\limits_{m\to\infty} D_i(m)/m = \lambda_i$ for some $\lambda_i \in [0, 1]$, where $\sum_{i\geq 0} \lambda_i = 1$, and moreover

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i\geq 1} i(i-2)D_i(m) = \sum_{i\geq 1} i(i-2)\lambda_i < \infty.$$

The limiting value λ_i, for each i, can be interpreted as the probability that a node picked at random has degree i. We assume that every asymptotic degree sequence $(\mathbf{d}(m))_{m\in Z^+}$ we work with is such that, for every m, the set of simple graphs that have $\mathbf{d}(m)$ as their degree sequence is non-empty. Additionally, we denote the generating polynomial of a sparse asymptotic degree distribution of a configuration model by $G_0(x) = \sum_{i=0}^{\infty} \lambda_i x^i$. Similar to the approach in [92] and [93], we will use generating functions to understand the site percolation mechanism on random graphs with a sparse asymptotic degree sequence.

8.1.1 Site percolation on $\mathcal{G}_m$

On a random graph with a sparse asymptotic degree sequence, $\mathcal{G}_m$, the site percolation mechanism marks a node as *open* with probability p or *closed* with probability $1-p$. An alternate viewpoint is to delete all edges corresponding to each vertex of $\mathcal{G}_m$ with probability $1-p$, independently of the other vertices in the graph. In other words, each vertex is turned into an isolated vertex (by removing all its edges) with probability $1-p$. Note that this leaves behind a random subgraph on m vertices which we denote by $\mathcal{G}'(m)$. Let $\mathcal{L}_1(\mathcal{G}'(m))$ be the lexicographically first component of $\mathcal{G}'(m)$ (this is the component that has maximum order and the smallest vertex it contains is smaller than the smallest vertex of every other component of maximum order; the comparison between the vertices is by means of the total ordering on V_m). We define

$$p_c = \sup\left\{ p \in [0,1] \;\middle|\; \frac{|\mathcal{L}_1(\mathcal{G}'(m))|}{m} \xrightarrow{\mathbb{P}} 0 \text{ as } m \to \infty \right\}.$$

Here $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, i.e., we say that $X_m \xrightarrow{\mathbb{P}} 0$ if for every $\epsilon > 0$ we have $\mathbb{P}(|X_m| > \epsilon) \to 0$ as $m \to \infty$. The convergence in probability is with respect to the sequence of probability spaces indexed by the set $\mathbb{Z}^+$, where for each m the probability is meant with respect to the graph $\mathcal{G}'(m)$.

From [90] and [94], it is known that if $\Delta \leq n^{1/4}$ and $G_0(x) = \sum_{i=0}^{\infty} \lambda_i x^i$ is sparse, then the critical probability can be expressed using the generating function as

$$p_c = \frac{G_0'(1)}{G_0''(1)}. \tag{8.1}$$

8.1.2 Relating site percolation and probabilistic forwarding

In the following, we consider probabilistic forwarding with $k = 1$ data packet. Recall that probabilistic forwarding corresponds to site percolation on the underlying graph but conditioned on the source being open. From our treatment of probabilistic forwarding on RGGs, it is clear that to quantify the expected number of transmissions of a single packet, it suffices to look at just the open nodes of the underlying graph. Denote by

$\mathcal{G}^+$ the sub-graph of $\mathcal{G}'(m)$ induced by the open nodes. Since the source is chosen at random independent of the probabilistic forwarding mechanism, we can first proceed by performing site percolation on $\mathcal{G}_m$ and then choosing a source from $\mathcal{G}^+$. Proceeding via this approach, we are interested in the expected size of the cluster of a randomly chosen vertex in $\mathcal{G}^+$ or the *typical cluster size*. Furthermore, notice that $\mathcal{L}_1(\mathcal{G}'(m)) = \mathcal{L}_1(\mathcal{G}^+)$, since the additional nodes in $\mathcal{G}'(m)$ are only the isolated nodes corresponding to the nodes which are closed. These do not contribute to the largest cluster.

In the super-critical region, i.e., $p > p_c$, there exists a giant component in $\mathcal{G}'(m)$ which contains εm nodes with high probability for large m, for some $\varepsilon > 0$. However, the exact dependence of ε on p is not known. Since the source is chosen at random, it is more likely to be in the largest cluster when operating in the super-critical regime. Thus, an estimate of the expected size of the largest cluster is crucial in characterizing probabilistic forwarding on the RRG. However, due to lack of accurate estimates, we leave this line of thought for future work.

Instead, we focus on the sub-critical region, $p < p_c$, where the size of the typical cluster is bounded. Recall that for a fixed k and δ, $p_{k,n,\delta} \to 0$ as $n \to \infty$ as proved in Section 4.3. For large n, the minimum forwarding probability $p_{k,n,\delta}$ falls below the percolation threshold, p_c. Our interest is to determine the expected cluster size of a randomly chosen vertex (call this C^+) in the sub-critical regime for the random graph $\mathcal{G}^+$. This has been derived in [92] and is given by

$$\mathbb{E}[|C^+|] = p\left[1 + \frac{pG_0'(1)}{1 - pG_1'(1)}\right], \tag{8.2}$$

where $G_1(x) = \dfrac{G_0'(x)}{G_0'(1)} = \dfrac{G_0'(x)}{\sum_i i\lambda_i}$. Since probabilistic forwarding corresponds to site percolation conditioned on the source being open, the expected number of transmissions is given by the expression in (8.2) divided by p. For n independent packet transmissions at probability $p_{k,n,\delta}$, we then obtain the expected total number of transmissions to be

$$\tau_{k,n,\delta} = n\left[1 + \frac{p_{k,n,\delta}\; G_0'(1)}{1 - p_{k,n,\delta}\; G_1'(1)}\right]. \tag{8.3}$$

8.2 Probabilistic forwarding on random regular graphs

In this section, we specialize the results of the previous section to random regular graphs and compare them with those obtained via simulations.

In a random d-regular graph, all vertices have degree d and hence $D_d(m) = |\{j \in V_n : d_j = d\}| = m$. Therefore, $\lambda_i = 0$ for all $i \neq d$ and $\lambda_d = 1$ resulting in the generating function of the asymptotic degree sequence $G_0(x) = x^d$. Moreover, the asymptotic degree sequence is sparse since

$$\sum_{i \geq 1} i(i-2)\lambda_i = d(d-2) < \infty.$$

Therefore, the results of the previous section can be applied for the case of random regular graphs (RRGs). Firstly, the critical probability for site percolation can be found using (8.1) to be

$$p_c = \frac{G_0'(1)}{G_0''(1)} = \frac{1}{d-1}.$$

The expected total number of transmissions while operating in the sub-critical region can be computed using (8.3) to be

$$\tau_{k,n,\delta} = n \ \frac{1 + p_{k,n,\delta}}{1 - (d-1)p_{k,n,\delta}} \tag{8.4}$$

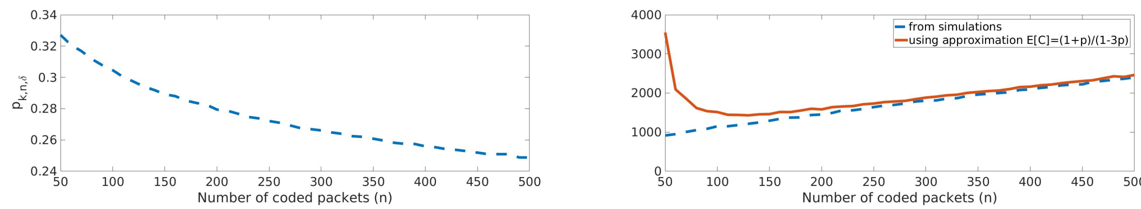
Notice that this does not depend on the number of nodes in $\mathcal{G}_m$.

Simulation results for probabilistic forwarding of a single packet ($k = 1$) on a random 4-regular graph on 1000 nodes are provided in Fig. 8.1. The number of coded packets, n, is varied from 20 to 500 so that we operate in the sub-critical region. For a random 4-regular graph, (8.4) simplifies to

$$\tau_{k,n,\delta} = n \ \frac{1 + p_{k,n,\delta}}{1 - 3 \ p_{k,n,\delta}}.$$

The total number of transmissions obtained using this is plotted alongside the simulation results in Fig. 8.1(b).

Notice that the curve obtained via (8.4) matches the simulations well for $n > 200$.



(a) Minimum retransmission probability (b) Expected total number of transmissions

Figure 8.1: Probabilistic forwarding of a single packet on a random 4-regular graph with 1000 nodes, in the sub-critical regime. The blue dashed curves are the simulations. The red curve in the second plot is obtained by substituting $p_{k,n,\delta}$ values from the simulations in (8.4).

For $n < 200$, the mismatch arises since the forwarding probability $p_{k,n,\delta}$ is close to the percolation threshold p_c, which for a random 4-regular graph is, $p_c = \frac{1}{d-1} = \frac{1}{3}$.

8.2.1 Conclusion and future work

The analysis of the probabilistic forwarding mechanism on random regular graphs was carried out in this chapter. The probabilistic forwarding mechanism was mapped to the site percolation on random graphs with a prescribed degree sequence. Results on the mean cluster size available in the literature were used to obtain the expected total number of transmissions in the sub-critical regime. These results not only indicate that the expected total number of transmissions required for a near broadcast increases for very large n, but also provide accurate estimates of the same.

It is to be noted that the analysis applies to general random graphs with a given degree sequence. Perhaps what would be more relevant is to obtain expressions for the expected number of transmissions and the fraction of successful receivers in the super-critical regime, which will explain the benefit to introducing coded packets with probabilistic forwarding as observed in the simulations (in Section 3.4). We leave this for future work.

Chapter 9

Summary and Future Work

In this thesis, we propose a novel probabilistic broadcast mechanism for ad-hoc networks and analyze it on three main network topologies: trees, grids and random geometric graphs. The proposed mechanism involves encoding k_s message packets into n coded packets at the source, such that reception of any k out of the n coded packets by a node in the network, suffices for that node to retrieve the data transmitted in the original k_s message packets. We are interested in the minimum probability ($p_{k,n,\delta}$) with which nodes in the network need to forward newly received packets so that, on the whole, a $1-\delta$ fraction of nodes are able to retrieve the k_s message packets from the source. Here, $\delta > 0$ is a small quantity. We termed this event a near-broadcast. The performance metric of interest is the expected total number of transmissions for a near-broadcast at the minimum forwarding probability, denoted as $\tau_{k,n,\delta}$.

Broadly speaking, on well-connected graphs such as grids, RGGs (in the super-critical region) and lattice structures, the expected total number of transmissions, $\tau_{k,n,\delta}$, decreases to a minimum and then gradually increases with the addition of coded packets, n. The decrease indicates energy savings compared to probabilistic forwarding with no coding. The value of the number of coded packets, n, and the value of the forwarding probability, $p_{k,n,\delta}$, corresponding to the minimum are optimal in terms of the energy expenditure for a near-broadcast. More specifically, the network when operated at these parameters has minimal expected number of transmissions while ensuring a near-broadcast. However, on trees, the expected total number of transmissions does not decrease with the addition of coded packets. In fact, it increases. This implies that introduction of coded packets along with the probabilistic forwarding protocol degrades the performance, since there are unnecessary transmissions of the additional coded packets.

While the qualitative performance of the probabilistic forwarding mechanism on different graphs is illustrative in its own right, the quantitative analysis on these topologies is of far greater value. Similar analytical techniques can be employed on other topologies or different variants of the probabilistic forwarding algorithm mentioned in Section 2.2. In this direction, the contributions of this thesis are listed below.

- The analysis of the probabilistic forwarding algorithm on binary trees involved basic concentration inequalities for random variables which have binomial distribution. These were honed to provide tight bounds for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$, thus showing that coding along with probabilistic forwarding on binary trees is not beneficial in terms of the expected number of transmissions required for a near-broadcast.

- On grids, the probabilistic forwarding mechanism was mapped to the site percolation process on $\mathbb{Z}^2$ and estimates for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$ were obtained in the limit as the size of the grid grows large. The workhorse of the analysis were the ergodic theorems for the site percolation process. Our analysis showed that introduction of coded packets helps to decrease the expected number of transmissions when compared to the scenario of no coding. The techniques for the grid extend easily to other lattice structures as well. In fact, the expressions for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$ can be directly used as long as one has knowledge of the percolation probability, $\theta(p)$, on these lattices.

- Characterizing the performance of the proposed broadcast mechanism on random geometric graphs was the holy grail of this thesis. Random graphs introduce multiple challenges which need to be addressed. Specifically, for RGGs, the connectedness of the underlying graph was justified by making suitable assumptions which were in line with the application we were interested in. Challenges arising from having a source node at the origin was addressed by using Palm theory in our analysis.

Ideas from the grid were built upon to provide an analytical framework. The probabilistic forwarding mechanism was modelled as a marked point process and ergodic theorems on them were employed to obtain estimates of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$. A heuristic approach was suggested based on mean-field assumptions to compare these estimates numerically with the simulation results. While this justifies the benefit of introducing coding with probabilistic forwarding, it also brings to the fore numerous open questions which need to be addressed.

- On random regular graphs, we obtained preliminary results on the performance of the probabilistic forwarding mechanism when the number of coded packets is large. Exploiting the tree-like structure of these graphs in the sub-critical regime and using generating functions, we were able to obtain good estimates of the expected number of transmissions, $\tau_{k,n,\delta}$, which were valid for large n. These techniques extend to general configuration models as well.

9.1 Future directions

In this section, we propose possible extensions to the work presented in this thesis.

9.1.1 Problems arising from the analysis

Our analysis of the probabilistic forwarding algorithm on both deterministic and random graphs present us with numerous problems. Some of these have been highlighted in the main text itself. We outline some others here. These problems are interesting in their own right and their understanding will benefit mathematicians and engineers alike.

- Broadly speaking, the theory of percolation presents innumerable questions which have not been answered for decades. These problems are easy to state but do not have clear answers. One such example from this thesis is the value of the site percolation threshold, p_c, on grids. The exact value of p_c is unknown even now, but large-scale Monte Carlo simulations indicate that $p_c \approx 0.593$. Perhaps, what is more relevant with respect to this thesis, is to obtain analytical expressions for the percolation probability $\theta(p)$ which can be used in our expressions for $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$.

- Similar concerns also arise in the field of continuum percolation where the critical intensity of the underlying point process, λ_c, is not exactly known. Additionally, the percolation probability $\theta(\lambda)$ does not admit any analytical expression.

- Probabilistic forwarding of n packets on the RGG gave rise to the term $\theta_{k,n}^{\text{ext}} = \mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C_{k,n}^{\text{ext}})$ in the expression for the expected fraction of successful receivers. An analytical expression for $\theta_{k,n}^{\text{ext}}$ in terms of $\theta^{\text{ext}}(\lambda, p)$ (which was the probability that the origin belongs to the IEC for site percolation on the RGG), would be useful in obtaining better estimates of $p_{k,n,\delta}$ and $\tau_{k,n,\delta}$. Perhaps, a simpler problem is to find the probability $\mathbb{P}^{\mathbf{0}}(\mathbf{0} \in C_{k,n}^{+})$. In terms of the marked point process formulation, for a point process $\Phi^{\mathbf{0}}$ with independent marks $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_n)$ where $Z_i \sim Ber(p)$, this is the probability that the origin is present in at least k out of the n infinite clusters. Each Z_i corresponds to a site percolation process on the underlying realization of the RGG. Conditional on the underlying RGG (or equivalently, Φ), the events corresponding to the presence of the origin in the infinite cluster of the ith and the jth percolation processes are independent. However, this is not true unconditionally. Intuitively, it is expected that the presence of the origin in the ith infinite cluster makes it more likely for it to be present in the jth infinite cluster as well. A mathematically rigorous understanding of this phenomenon is necessary.

- As an extension of the techniques presented here, one could consider each communication link between nodes to be noisy. Then, even though a node might forward a packet with probability p, it will be received only by a subset of its neighbours depending on the packet drop probability, q, induced by the noisy channel. This can be modelled as simultaneous bond and site percolation on the underlying graph. For lattices, this corresponds to site percolation on a random subset of the lattice.

9.1.2 Quantifying performance on different graphs

In this section, we try to provide qualitative and quantitative justifications for the difference in behaviour of the probabilistic forwarding protocol due to different underlying network topologies. We have already seen a comparison of the tree and grid graphs in Section 6.6.3. We saw that the difference in behaviour can be attributed to multiple paths from the source to a network node. Presence of multiple paths between pairs of vertices of a graph is an indication of the connectedness of a graph. In this section, we explore a popular metric, the conductance of the graph, which is widely used to quantify the connectedness of a graph to explain the probabilistic forwarding mechanism. Graph conductance has been a popular metric to quantify how well-connected a graph is. While its predominant use has been in bounding the mixing time of Markov chains, it has also been used to characterize rumour spreading on graphs in [95] and [96].

Definition 4. *Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let $S \subset V$, and $S^c = V \backslash S$. Let d_u denote the degree of vertex $u \in S$, and $|A|$ denote the cardinality of set A. Then the conductance $\Phi(S, S^c)$ is given by*

$$\Phi(S, S^c) = \frac{|E_{S,S^c}|}{\sum_{u \in S} d_u},$$

where $E_{S,S^c} = \{(u, v) \in E \mid u \in S, v \in S^c\}$. The conductance of the graph G is defined as

$$\Phi_G = \min_{|S| \leq |V|/2} \Phi(S, S^c).$$

A small value of $\Phi(S, S^c)$ indicates that there are lot more connections between vertices within S than between vertices within S and S^c. Naturally, if the source is present within S, it is hard for the information to reach S^c.

Finding the conductance for any graph boils down to finding a large subset of vertices S with minimum edges between S and S^c. For a complete graph with N vertices, K_N, the best partitioning occurs when the graphs vertices are partitioned into two equal halves, and it has conductance $\Phi_{K_N} = \frac{1}{2}\left(1 + \frac{1}{N-1}\right) = \Theta(1)$. For a cycle of N vertices, C_N, the smallest conductance partitioning occurs when its left and right halves are separate

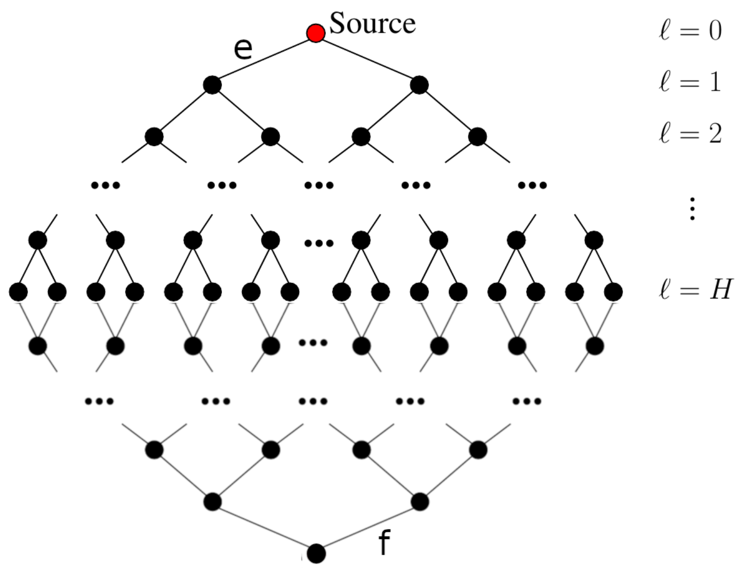


Figure 9.1: A graph with conductance similar to a binary tree but whose behaviour to probabilistic forwarding is grid-like.

partitions, and it has conductance $\Phi_{C_N} = \frac{2}{N}$. In an intuitive sense this means that unlike a cycle, a complete graph is not partitionable. Further, for a d-dimensional hypercube with 2^d vertices, the partitioning set $S = \{x \in \{0,1\}^d | x_1 = 0\}$ minimizes the conductance of the graph. The conductance can be computed to be $\frac{1}{d}$ or equivalently $\frac{1}{\log_2(N)} = \Theta\left(\frac{1}{\log_2(N)}\right)$ (see e.g., [97]).

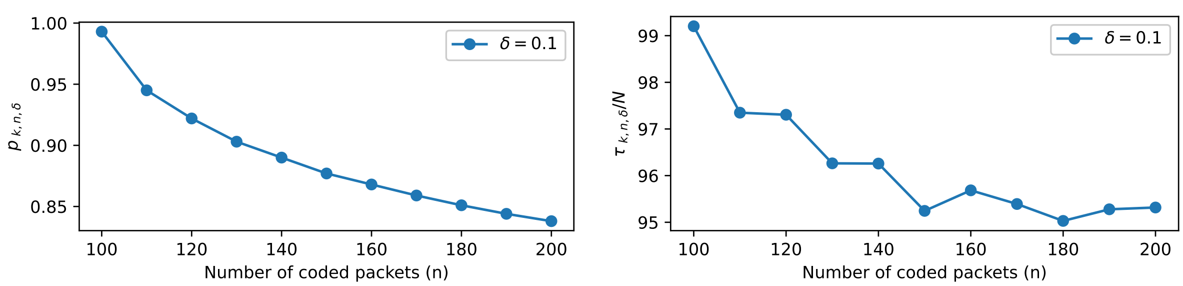
For the binary tree of height H with $N = 2^{H+1} - 1$ vertices, the partitioning set S is one of the subtrees at the root. This gives a conductance of $\frac{1}{N-2} = \Theta(1/N)$ for the binary tree. Similarly for the square grid containing N vertices, the separation with minimum edges across it is a line parallel to one of the axes. The conductance of the grid is then given by (see e.g., [98]) $\left(2N^{\frac{1}{2}}\right)^{-1} = \Theta(1/\sqrt{N})$. .

This suggests that conductance can be used as a quantitative metric to distinguish different behaviour – tree-like or grid-like – for the probabilistic forwarding mechanism. Bolstering this intuition further is the observation that if edges are removed from the square grid as in Section 3.4 to obtain the graph G_{15}, the conductance computed on this graph $\Phi_{G_{15}} = \Theta(1/N)$. Indeed, notice that G_{15} shows a tree-like behaviour (Fig. 3.13) for probabilistic forwarding mechanism with coded packets.

However, this intuition is not completely valid. A counterexample can be obtained by gluing together two binary trees at the leaves as shown in Fig. 9.1. The conductance of this graph can be computed using the cut which passes only through the edges e and f in the figure and comprises of half the number of vertices.

$$\Phi(S, S^c) = \frac{2}{N} = \Theta(1/N).$$

This is similar to the conductance of a tree. However, simulations of the probabilistic forwarding algorithm on this graph indicate that the expected total number of transmissions on this graph decreases with the addition of coded packets. Simulations with $k = 100$ packets on this graph with $H = 9$ and $\delta = 0.1$ are shown in Fig. 9.2. This behaviour is akin to that of a grid.



(a) Minimum retransmission probability (b) Expected total number of transmissions

Figure 9.2: Simulations on the graph shown in Fig. 9.1. Probabilistic forwarding done with $k = 100$ packets.

A possible reason for this mismatch could be that conductance inherently is a good measure of 'bottlenecks' in the graph. However, from our discussion in the previous subsection, it is the number of different paths between two vertices which produces the specific behaviour for probabilistic forwarding. This is not being captured. Nevertheless, a higher conductance implies a well-connected underlying graph. One could possibly exploit this to obtain a sufficient condition for a graph to show grid-like behaviour during the probabilistic forwarding mechanism. In this thesis, we do not explore this any further, but leave it for future work. However, based on the simulations, we make the following conjecture.

Conjecture 9.1.1. *Consider the probabilistic forwarding mechanism with k message packets being encoded into n coded packets on a graph G of N nodes such that the conductance $\Phi_G \geq \frac{c}{\sqrt{N}}$, for some constant $c > 0$. Then the expected total number of transmissions at the minimum forwarding probability for a near-broadcast ($p_{k,n,\delta}$), shows a decrease (initially) with the addition of coded packets. Moreover, for a carefully chosen value of n and $p_{k,n,\delta}$, the expected number of transmissions is minimized.*

Another aspect that conductance does not capture is the location of the source node in the graph. Different choices of the source can result in variations in the behaviour of the probabilistic forwarding protocol. Global properties such as conductance are more likely to be reliable metrics for characterizing probabilistic forwarding on graphs where the choice of the source node does not affect the broadcast mechanism too much. This requirement is satisfied, for example, by vertex-transitive graphs. These are graphs G in which for every two vertices v_1 and v_2 of G, there exists an automorphism $f : V(G) \to V(G)$ such that $f(v_1) = v_2$. Loosely speaking, this means that the graph G looks exactly the same from the perspective of any vertex. Cayley graphs — which include as special cases complete graphs, N-cycles, and hypercubes — provide a broad class of examples of vertex-transitive graphs. On such graphs, pathological behaviour caused by a bad choice of the source node is not possible. Thus, a global measure of connectedness, such as conductance, may be a reliable predictor of the performance of probabilistic forwarding with coded packets.

9.1.3 Algorithm extension

In this section, we discuss some directions for future work for probabilistic forwarding with coding on deterministic graphs. A plausible extension to the probabilistic forwarding mechanism proposed in 2.2 is when the nodes in the network decide to forward a packet with different probabilities based on their distance from the source. We formulate this problem here on binary trees.

On a binary tree, the nodes at a particular level ℓ are identical. It only makes sense to assume that all of them decide to transmit the packet with the same probability. Let the nodes at level ℓ of a binary tree transmit a packet with probability p_ℓ for $\ell \in \{0, 1, \cdots, H\}$.

As before, $p_0 = 1$ since the source transmits all the packets. The probability that a node at level $\ell \in \{1, \cdots, H\}$ receives a packet is $\prod_{i=0}^{\ell-1} p_i \triangleq r_\ell$. Note that $r_1 = p_0 = 1$. Additionally

$$\begin{aligned}\mathbb{P}(\text{node } \mathbf{v} \text{ at level } \ell \text{ receives at least } k \text{ out of } n \text{ packets}) &= \sum_{j=k}^{n} \binom{n}{j} r_\ell^j (1-r_\ell)^{n-j} \\ &= \mathbb{P}(Z_\ell \geq k),\end{aligned}$$

where $Z_\ell \sim \text{Bin}(n, \prod_{i=1}^{\ell-1} p_i)$. The expected number of successful receivers ($R_{k,n}$) is then given by

$$\mathbb{E}[R_{k,n}] \;=\; 1 + \sum_{\ell=1}^{H} 2^\ell \, \mathbb{P}(Z_\ell \geq k). \tag{9.1}$$

Likewise, a node at level ℓ receives and transmits a packet with probability $r_\ell p_\ell = r_{\ell+1}$. The expected total number of transmissions for n packets is then $n(1 + \sum_{\ell=1}^{H} 2^\ell r_{\ell+1})$.

We wish to find the set of forwarding probabilities which ensure a near-broadcast. Similar to (5.3), the condition for a near-broadcast reduces to

$$\frac{\sum_{\ell=0}^{H-1} 2^{\ell+1} \mathbb{P}(Z_{\ell+1} \leq k-1)}{2^{H+1}-1} \leq \delta.$$

The set of probabilities $\{\mathbf{p} = (p_\ell), \ell = 1, 2, \cdots, H\}$ which satisfy the above equation constitute the feasible set to ensure a near-broadcast. Our interest is in those feasible forwarding probabilities, which minimizes the total number of transmissions while ensuring a near-broadcast. This can be formulated as an optimization problem:

$$\begin{aligned}&\underset{\mathbf{p}}{\text{minimize}} \quad n + n\sum_{\ell=1}^{H} 2^\ell r_{\ell+1} \\ &\text{subject to} \quad \sum_{\ell=0}^{H-1} \frac{2^{\ell+1}}{2^{H+1}-1} \mathbb{P}(Z_{\ell+1} \leq k-1) \leq \delta, \\ &\qquad\qquad\qquad\qquad \text{where} \;\; Z_\ell \sim \text{Bin}(n, r_\ell).\end{aligned}$$

This can be further reduced as follows. Since the minimization does not depend on n, it can be removed from the objective function. This is equivalent to minimizing the number of transmissions of a single packet. Additionally, if it is assumed that the network nodes possess knowledge of the height of the tree, then the nodes at level H need not transmit the packet, i.e., $p_H = 0$. We then have the following alternate problem:

$$\begin{aligned}
\underset{\mathbf{r}=(r_2,\cdots,r_H)}{\text{minimize}} \quad & 1 + 2r_2 + 4r_3 + \cdots + 2^{H-1} r_H \\
\text{subject to} \quad & 0 \le r_i \le 1, \text{ for } i = 2, \cdots, H, \\
& r_2 \ge r_3 \ge \cdots \ge r_H, \\
& \sum_{\ell=0}^{H-1} \frac{2^{\ell+1}}{2^{H+1}-1} \mathbb{P}(Z_{\ell+1} \le k-1) \le \delta, \\
& \text{where} \;\; Z_\ell \sim \mathrm{Bin}(n, r_\ell).
\end{aligned}$$

We do not attempt to solve this problem here, but leave it as future work.

While this seems like a reasonable extension to the probabilistic forwarding protocol, nodes in the network need to have additional information related to their relative distance from the source. On a tree, this information is easy to procure from the source. A counter can be added to the packet header which is updated as it traverses down the tree. Once the packet reaches a node, the header is decoded to determine the level at which the node is present, and the corresponding forwarding probability is used to transmit it further down the tree.

However on other graph structures, such as grids, a node might receive the packet from a long convoluted path. Determining the distance of the node from the source and the forwarding probability to be used is non-trivial on such topologies.

9.1.4 Mobility

The probabilistic forwarding mechanism with coded packets is a completely decentralized and distributed algorithm. This makes it amenable to be deployed on mobile ad-hoc networks (MANETs) or vehicular networks (VANETs) where the individual nodes are moving. Moreover, simulation studies indicate that mobility improves connectivity in such networks (see e.g., [99–102]). The authors in [103] collect the numerous mobility models that have been discussed in the literature for ad-hoc networks. Among these, the random waypoint model is worth a mention since it is analytically tractable (see [104]). In this model, nodes choose a random destination point within a prescribed area, and a random speed in a prescribed range. When they reach the destination, they remain static for a predefined pause time and then start moving again according to the same rule. It is known that the spatial distribution of network nodes moving according to this model is, in general, nonuniform. The approximate stationary distribution has been obtained in [104]. It would be interesting to characterize the performance of our probabilistic broadcast algorithm in conjunction with such mobility models.

Appendices

Appendix A

Auxiliary results for Part II

In this appendix, we collect the auxiliary results which have been used in the analysis of the probabilistic forwarding mechanism on trees and grids.

A.1 Coupling argument

Lemma A.1.1. *For $\zeta \sim Bin(n,p)$ and $0 \leq k \leq n$, $\mathbb{P}(\zeta \geq k)$ is a continuous, monotonically increasing function of p.*

Proof. Note first that $\mathbb{P}(\zeta \geq k) = \sum_{k' \geq k} \binom{n}{k} p^{k'} (1-p)^{n-k'}$, which is a polynomial in p, and hence $\mathbb{P}(\zeta \geq k)$ is continuous in p. Monotonicity is by a standard coupling argument: Let U_i, $i = 1, 2, \ldots, n$, be i.i.d. Unif$[0,1]$ random variables. For $p \leq p'$, let $X_i = \mathbb{I}_{\{U_i \leq p\}}$ and $X_i' = \mathbb{I}_{\{U_i \leq p'\}}$, so that the X_is are i.i.d. Ber(p) and the X_i's are i.i.d Ber(p'). Then, $\zeta = \sum_{i=1}^n X_i$ is Bin(n,p), while $\zeta' = \sum_{i=1}^n X_i'$ is Bin(n,p'). By construction, $X_i(U_i) \leq X_i'(U_i)$, and hence, $\zeta \leq \zeta'$ almost surely. Thus, $\mathbb{P}(\zeta \geq k) \leq \mathbb{P}(\zeta' \geq k)$. □

A.2 Bounds for the CDF of a Binomial random variable

The following theorem from [105] gives tight bounds on the CDF of a binomial random variable in terms of the standard normal CDF.

Theorem A.2.1 ([105], Theorem 1). *Let $0 \leq x, p \leq 1$ and define $D(x \,||\, p) := x \ln \frac{x}{p} + (1-x)\ln\frac{1-x}{1-p}$, $\operatorname{sgn}(x) := \frac{x}{|x|}$ for $x \neq 0$, and $\operatorname{sgn}(0) := 0$. Let $\{C_{n,p}(k)\}_{k=0}^{n}$ be defined as follows:*

$$C_{n,p}(0) = (1-p)^n, \; C_{n,p}(n) = 1 - p^n,$$

$$C_{n,p}(k) = \Phi\left(\operatorname{sgn}\left(\frac{k}{n} - p\right)\sqrt{2nD\left(\frac{k}{n} \,||\, p\right)}\right), \; 1 \leq k < n.$$

For a binomial random variable $X \sim \operatorname{Bin}(n,p)$, for every $k = 0, 1, ..., n-1$, and for every $p \in (0,1)$,

$$C_{n,p}(k) \leq \mathbb{P}(X \leq k) \leq C_{n,p}(k+1).$$

Equalities hold for $k = 0$ and $k = n-1$ only.

A.3 FKG inequality

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Ω be equipped with a partial order $\leq$. We say that an event A in $\mathcal{F}$ is increasing if $\mathbf{1}_A(\omega) \leq \mathbf{1}_A(\omega')$ whenever $\omega \leq \omega'$, where $\mathbf{1}_A$ is the indicator function of A. We call A decreasing if its complement A^c is increasing. More generally, a random variable X is called increasing if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$; X is called decreasing if $-X$ is increasing.

Theorem A.3.1 (Fortuin, Kasteleyn, and Ginibre (1971), Harris (1960)). *(see [80, Chapter 2])*

- *If X and Y are increasing random variables such that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, then*

$$\mathbb{E}[XY] \geq \mathbb{E}[X]\;\mathbb{E}[Y].$$

- *If A and B are increasing events, then*

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\;\mathbb{P}(B).$$

Similar inequalities are valid for decreasing random variables and events. For example, if X and Y are both decreasing then $-X$ and $-Y$ are increasing, giving that $\mathbb{E}[XY] \geq \mathbb{E}[X]\ \mathbb{E}[Y]$, so long as X and Y have finite second moments. Similarly, if X is increasing and Y is decreasing, then we may apply the FKG inequality to X and $-Y$ to find that $\mathbb{E}[XY] \leq \mathbb{E}[X]\ \mathbb{E}[Y]$.

As an example, consider site percolation on $\mathbb{Z}^2$. Here $\Omega = \{0,1\}^{\mathbb{Z}^2}$ and the partial order is defined as $\omega_1 \leq \omega_2$ if $\omega_1(v) \leq \omega_2(v)$ for all $v \in \mathbb{Z}^2$. A vertex u is said to be open if $\omega(u) = 1$. Let $u, v \in \mathbb{Z}^2$, and recall the definition $u \longleftrightarrow v$, which meant that u and v are connected by a path of open vertices. The event $\{\mathbf{0} \longleftrightarrow u\}$ and the random variable measuring the number of different paths between $\mathbf{0}$ and u are increasing.

Let $\{u \longleftrightarrow \infty\}$ denote the event that u is in the infinite cluster. Then

$$\begin{aligned}\mathbb{P}(\{u \longleftrightarrow \infty\}) &\geq \mathbb{P}(\{u \longleftrightarrow v\} \cap \{v \longleftrightarrow \infty\}) \\ &\overset{FKG}{\geq} \mathbb{P}(\{u \longleftrightarrow v\})\ \ \mathbb{P}(\{v \longleftrightarrow \infty\}).\end{aligned}$$

Similarly, we can repeat the same calculation with u and v interchanged. From this, we can deduce that if $\mathbb{P}(\{u \longleftrightarrow \infty\}) > 0$, then $\mathbb{P}(\{v \longleftrightarrow \infty\}) > 0$ and vice versa. Thus, it does not make a difference whether one defines the critical probability for site percolation as $p_c = \inf\{p \mid \mathbb{P}(\{u \longleftrightarrow \infty\}) > 0\}$ or $p_c = \inf\{p \mid \mathbb{P}(\{v \longleftrightarrow \infty\}) > 0\}$. Owing to the translation invariance of the $\mathbb{Z}^2$ lattice and the probability measure $\mathbb{P}$, it is customary to take the percolation probability $\theta(p) = \mathbb{P}(\mathbf{0} \in C) = \mathbb{P}(\mathbf{0} \longleftrightarrow \infty)$ giving $p_c = \inf\{p \mid \theta(p) > 0\}$.

Appendix B

Auxiliary results for Part III

In this appendix, we collect the auxiliary results which have been used in the analysis of the probabilistic forwarding mechanism on random geometric graphs.

B.1 Palm probabilities

In this section, we prove three main propositions which will be used in the analysis of the probabilistic forwarding protocol. Let $\mathcal{G} \sim RGG(\lambda, 1)$ be a random geometric graph on $\mathbb{R}^2$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The underlying Poisson point process, Φ, is of intensity λ. The intensity λ is such that we operate in the super-critical region, i.e., $\lambda > \lambda_c$. Let $C \equiv C(\Phi)$ be the unique infinite cluster in $\mathcal{G}$. Let $\Phi^{\mathbf{0}} = \Phi \cup \{\mathbf{0}\}$ denote the Palm version of Φ and let $C(\Phi^{\mathbf{0}})$ be the infinite cluster in it. Denote by $\mathbb{P}^{\mathbf{0}}$, the Palm probability of the origin and $\mathbb{E}^{\mathbf{0}}$, the expectation with respect to $\mathbb{P}^{\mathbf{0}}$. We now have the following proposition relating the expected value with respect to $\mathbb{E}$ and $\mathbb{E}^{\mathbf{0}}$ of the fraction of vertices in C within Γ_m.

Proposition B.1.1.

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{|C \cap \Gamma_m|}{m^2}\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{|C \cap \Gamma_m|}{m^2}\right]$$

Proof. Let $C_1, C_2, \cdots, C_K$ be finite components in $\mathcal{G}$ which intersect the ball of radius 1 centered at the origin, i.e., $C_i \cap B_1(\mathbf{0}) \neq \emptyset, \quad \forall i \in \{1, 2, \cdots, K\}$. Since vertices from distinct finite components C_i and C_j, should be at least at a distance of 1 from each other, the number of such components is bounded. In particular, K is a random variable with $K \leq 7$ *a.s.*. The infinite clusters in the $RGG(\Phi^{\mathbf{0}}, 1)$ and $RGG(\Phi, 1)$ models can be related in the following way:

$$C(\Phi^{\mathbf{0}}) = \begin{cases} C(\Phi) \cup C_1 \cup \cdots \cup C_K \cup \{\mathbf{0}\} & \text{if } C \cap B_1(\mathbf{0}) \neq \emptyset \\ C(\Phi) & \text{if } C \cap B_1(\mathbf{0}) = \emptyset \end{cases}$$

Using this, we can write

$$\frac{|C(\Phi^{\mathbf{0}}) \cap \Gamma_m|}{m^2} = \frac{|C(\Phi) \cap \Gamma_m|}{m^2} + \sum_{i=1}^{K} \frac{|C_i \cap \Gamma_m|}{m^2} \mathbb{1}\{C \cap B_1(\mathbf{0}) \neq \emptyset\}$$

Since $K \leq 7$ *a.s.* and $|C_i| < \infty$ for all $i = 1, 2, \cdots, K$, we have

$$\sum_{i=1}^{K} \frac{|C_i \cap \Gamma_m|}{m^2} \xrightarrow{m \to \infty} 0 \quad \mathbb{P}\text{-a.s.}.$$

Thus, we deduce that

$$\lim_{m \to \infty} \frac{|C(\Phi^{\mathbf{0}}) \cap \Gamma_m|}{m^2} = \lim_{m \to \infty} \frac{|C(\Phi) \cap \Gamma_m|}{m^2} \qquad \mathbb{P}\text{-a.s.} \tag{B.1}$$

Since the random variables involved are bounded by 1, applying the dominated convergence theorem (DCT) gives the desired result. □

Corollary B.1.2.

$$\lim_{m \to \infty} \mathbb{E}^{\mathbf{0}} \left[\frac{|C \cap \Gamma_m|}{m^2} \right] = \lambda \theta(\lambda)$$

Proof. This directly follows from the previous proposition and (7.7). □

Next, consider the formulation of the marked point process described in Section 7.3.

Let $C^{\text{ext}} \equiv C^{\text{ext}}(\Phi)$ be the infinite extended cluster (IEC). We now show the following proposition relating C^{ext} and the Palm version of C^{ext}.

Proposition B.1.3.

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{|C^{ext}\cap\Gamma_m|}{m^2}\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{|C^{ext}\cap\Gamma_m|}{m^2}\right]$$

Proof. The proof is along the same lines as that in Proposition B.1.1. Let $C_1, C_2, \cdots, C_K$ be finite components in $\mathcal{G}^+$ which intersect the ball of radius 1 centered at the origin, i.e., $C_i \cap B_1(\mathbf{0}) \neq \emptyset, \quad \forall i \in \{1, 2, \cdots, K\}$. Here again $K \leq 7$ *a.s.*. Now, suppose that $C^+ \cap B_1(\mathbf{0}) \neq \emptyset$, then regardless of the mark of the origin, it is true that $C^{\text{ext}}(\Phi^{\mathbf{0}}) \subseteq C^{\text{ext}}(\Phi) \cup C_1^{\text{ext}} \cup \cdots \cup C_K^{\text{ext}}$ (with equality being true when the origin has mark 1). If on the other hand $C^+ \cap B_1(\mathbf{0}) = \emptyset$, then $C^{\text{ext}}(\Phi^{\mathbf{0}}) = C^{\text{ext}}(\Phi)$. Using this, we can write

$$\begin{aligned}\frac{|C^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2} \leq{}& \frac{|C^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2}\\ &+ \sum_{i=1}^{K}\frac{|C_i^{\text{ext}}\cap\Gamma_m|}{m^2}\ \mathbb{1}\{C^+\cap B_1(\mathbf{0})\neq\emptyset\}.\end{aligned}$$

Note that, if C_i is a finite cluster, then so is C_i^{ext} and hence the summation on the RHS above tends to 0 as $m \to \infty$. Since we trivially have that

$$\frac{|C^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2} \leq \frac{|C^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2},$$

in the limit of large m, the fraction $\frac{|C^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2}$ is sandwiched between the two limits yielding

$$\lim_{m\to\infty}\frac{|C^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2} = \lim_{m\to\infty}\frac{|C^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2} \qquad \mathbb{P}\text{-a.s.}$$

Using DCT gives the statement of the proposition. $\square$

A similar argument extends to $C_{k,n}^{\text{ext}}$ as well, which is stated in the following proposition.

Proposition B.1.4.

$$\lim_{m\to\infty} \mathbb{E}^{\mathbf{0}}\left[\frac{|C_{k,n}^{ext}\cap\Gamma_m|}{m^2}\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{|C_{k,n}^{ext}\cap\Gamma_m|}{m^2}\right]$$

Proof. Firstly, note that

$$\frac{|C_{k,n}^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2} \geq \frac{|C_{k,n}^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2}. \tag{B.2}$$

The nodes in $C_{k,n}^{\text{ext}}(\Phi^{\mathbf{0}})$ can be related to those in $C_{k,n}^{\text{ext}}(\Phi)$ in the following way. Let $C_1^+, C_2^+, \cdots, C_n^+$ denote the infinite clusters corresponding to each of the n packets and let $C_{i,1}, C_{i,2}, \cdots, C_{i,K_i}$ denote the finite clusters corresponding to the $i-$th packet which intersect the ball of radius 1 at the origin. Here again, $K_i \leq 7$ *a.s.* for all i. Proceeding with similar reasoning as that of Proposition B.1.3, we can obtain

$$\frac{|C_{k,n}^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2} \leq \frac{|C_{k,n}^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2} + \sum_{\substack{i\in[n]\\ C_i^+\cap B_1(\mathbf{0})\neq\emptyset}} \sum_{j=1}^{K_i} \frac{|C_{i,j}^{\text{ext}}\cap\Gamma_m|}{m^2} \tag{B.3}$$

The summation on the RHS is a finite sum with at most $7n$ terms with each term consisting of fraction of nodes in some finite cluster. By taking limits as $m\to\infty$, this fraction vanishes. Therefore the fraction $\frac{|C_{k,n}^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2}$ is sandwiched between the two limits in (B.2) and (B.3) yielding

$$\lim_{m\to\infty}\frac{|C_{k,n}^{\text{ext}}(\Phi^{\mathbf{0}})\cap\Gamma_m|}{m^2} = \lim_{m\to\infty}\frac{|C_{k,n}^{\text{ext}}(\Phi)\cap\Gamma_m|}{m^2} \qquad \mathbb{P}\text{-a.s.}$$

Using DCT gives the statement of the proposition. ∎

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