Community detection on multilayer hypergraphs using the aggregate similarity matrix

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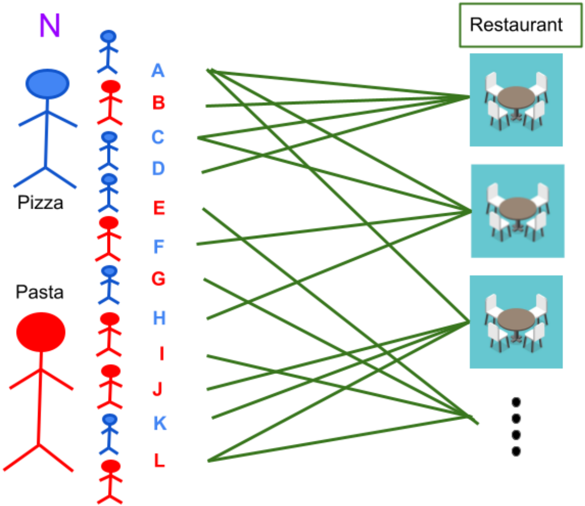
Motivating example:



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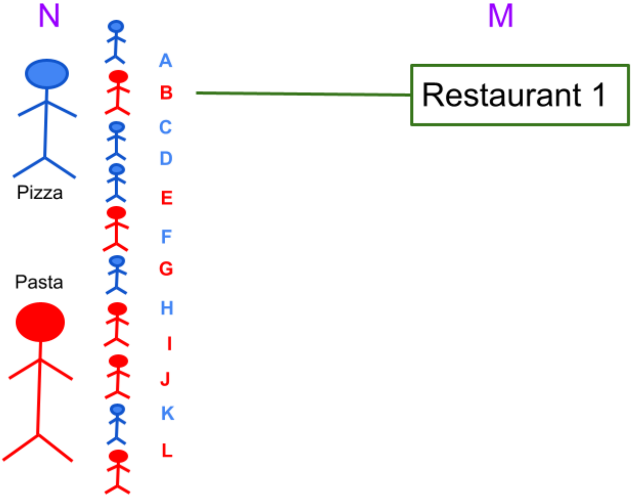

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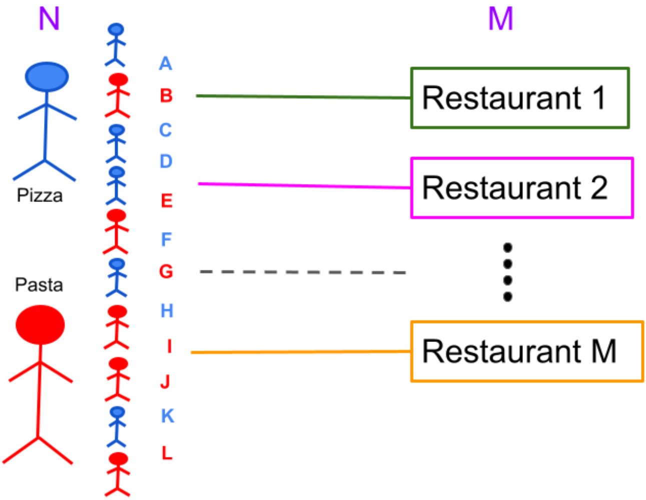

Multilayer hypergraph

Multilayer HSBM

- N Vertices - $\{1, \cdots, N\} =: [N]$. Two communities $\{-1, +1\}$.
- M Layers - $\{1, \cdots, M\}$ indexed by m
- d vertices in every hyperedge

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Step 1: Sample the communities

$$\boldsymbol{\sigma} \sim \text{Unif}\left(\left\{\sigma \in \{\pm 1\}^N \;\middle|\; \text{equal number of } +1 \text{ and } -1\right\}\right)$$

Step 2: For each layer $m \in \{1, \cdots, M\}$ and for each hyperedge $e \subset [N]$ with $|e| = d$, set

$$A_e^{(m)} = \begin{cases} 1 & \text{with prob. } p_e^{(m)} & (e \text{ is present in layer } m) \\ 0 & \text{with prob. } 1 - p_e^{(m)} & (e \text{ is not present in layer } m) \end{cases},$$

Hypergraph incidence matrix - $\mathbf{A} = (A_e^{(m)})$

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Multilayer HSBM: Specifications

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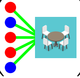
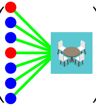
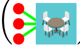
1. Community profile of hyperedge e denoted $\boldsymbol{\tau} \equiv (\boldsymbol{\tau}(e))$

$\boldsymbol{\tau}$ () $= (3, 2)$, $\boldsymbol{\tau}$ () $= (2, 5)$, $\boldsymbol{\tau}$ () $= (3, 0)$

$$p_e^{(m)} = p_{\boldsymbol{\tau}(e)}^{(m)}$$

For two communities, $\boldsymbol{\tau}(e) \in \{(0, d), (1, d-1), \cdots, (d, 0)\}$.

2. Scaling regime: For an edge with community profile $\boldsymbol{\tau}(e)$,

$$p_{\boldsymbol{\tau}(e)}^{(m)} = \alpha_{\boldsymbol{\tau}(e)}^{(m)} \frac{\log N}{\binom{N-1}{d-1}}.$$

3. Symmetricity:

$$\alpha_{(r,d-r)}^{(m)} = \alpha_{(d-r,r)}^{(m)}$$

Homogeneous: $\alpha_{\boldsymbol{\tau}} = \alpha$ if $\boldsymbol{\tau} \in \{(d, 0), (0, d)\}$; else $\alpha_{\boldsymbol{\tau}} = \beta$.

Gaudio, J. and Joshi, N., 2022. Community detection in the hypergraph sbm: Optimal recovery given the similarity matrix. arXiv preprint arXiv:2208.12227.

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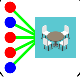
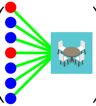
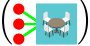
$\boldsymbol{\tau}\Big($  $\Big) = (3,2)$, $\boldsymbol{\tau}\Big($  $\Big) = (2,5)$, $\boldsymbol{\tau}\Big($  $\Big) = (3,0)$

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Assortativity

$\alpha_{({\color{red}\bullet\bullet\bullet\bullet\bullet})}$ $\alpha_{({\color{red}\bullet\bullet\bullet\bullet}{\color{blue}\bullet})}$ $\alpha_{({\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet\bullet}{\color{blue}\bullet})}$

Assortativity

Assortative:

$$\alpha_{({\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet})} \quad > \quad \alpha_{({\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet})} \quad > \quad \alpha_{({\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet})}$$

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\alpha_{(\color{red}{\bullet\bullet\bullet\bullet\bullet})} & > & \alpha_{(\color{red}{\bullet\bullet\bullet\bullet}\color{blue}{\bullet})} & > & \alpha_{(\color{red}{\bullet}\color{blue}{\bullet}\color{red}{\bullet\bullet}\color{blue}{\bullet})} \\
\shortparallel & & \shortparallel & & \shortparallel \\
\alpha_{(\color{blue}{\bullet\bullet\bullet\bullet\bullet})} & > & \alpha_{(\color{red}{\bullet}\color{blue}{\bullet\bullet\bullet\bullet})} & > & \alpha_{(\color{blue}{\bullet\bullet}\color{red}{\bullet\bullet}\color{blue}{\bullet})}
\end{array}
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$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

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Disassortative:

$$\alpha_{({\color{red}\bullet\bullet\bullet\bullet\bullet})} < \alpha_{({\color{red}\bullet\bullet\bullet\bullet}{\color{blue}\bullet})} < \alpha_{({\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet\bullet}{\color{blue}\bullet})}$$

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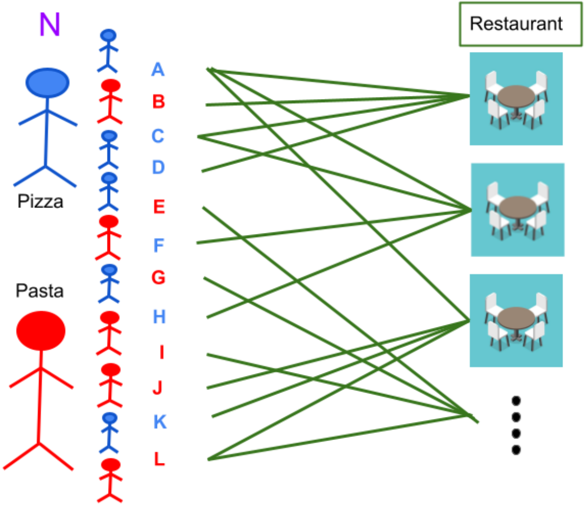
$$

- Each layer could be either assortative or disassortative.
- Define assortativity

$$
\xi \ := \ \sum_{m=1}^{M} \sum_{r=0}^{d-1} \binom{d-1}{r} (d-1-2r)\alpha_{(r,d-r)}^{(m)}.
$$

- Assortative: $\xi > 0$ and Disassortative: $\xi < 0$.
- For $d = 5$, $\quad \xi = 4\alpha_{(0,5)} + 4\alpha_{(1,4)} - 8\alpha_{(2,3)}$.

Motivation



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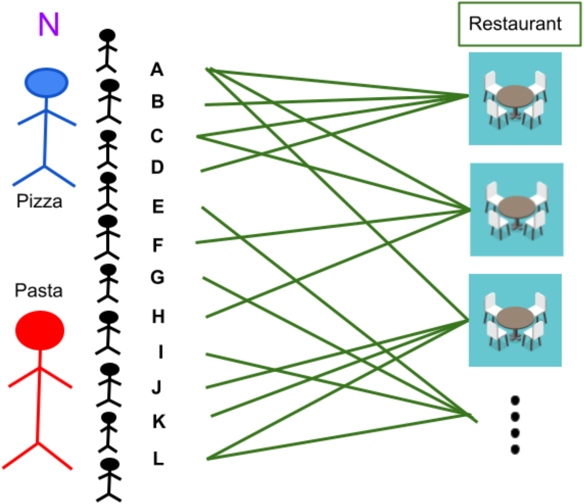

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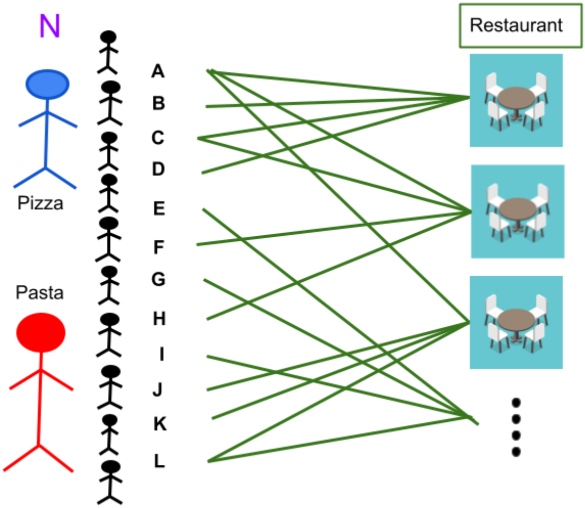
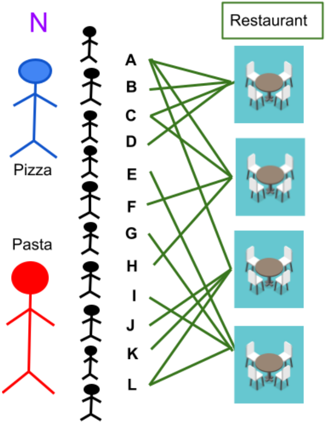


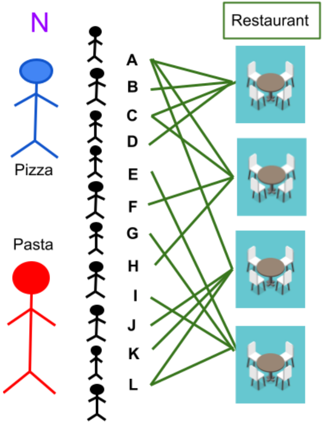
Recover communities while maintaining privacy.

Similarity matrix



	A	B	C	D	E	F	G	H	I	J	K	L
A	0											
B		0										
C			0									
D				0								
E					0							
F						0						
G							0					
H								0				
I									0			
J										0		
K											0	
L												0

Similarity matrix



	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	2	1	0	1	0	1	0	1	1	1
B		0	1	1	0	0	0	0	0	0	0	0
C			0	1	0	1	0	1	0	0	0	0
D				0	0	0	0	0	0	0	0	0
E					0	0	1	0	1	0	0	1
F						0	0	1	0	0	0	0
G							0	0	1	0	0	1
H								0	0	0	0	0
I									0	0	0	1
J										0	0	0
K											0	0
L												0

Problem formulation

Data: $(\boldsymbol{\sigma}, \mathbf{A}) \sim \mathrm{HSBM}(N, M, d, (\alpha_{\boldsymbol{\tau}}^{(m)}))$

Given: $N \times N$ aggregate similarity matrix $\mathbf{W} = (\mathbf{W}_{ij})$, such that

$$\mathbf{W}_{ij} = \sum_{m=1}^{M} W_{ij}^{(m)},$$

where

$$\begin{aligned} W_{ij}^{(m)} &= \#\text{ of hyperedges that contain both } i \text{ and } j \text{ in layer } m \\ &= \sum_{e: e \ni i,j} A_e^{(m)} \end{aligned}$$

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Want to find an estimate $\hat{\boldsymbol{\sigma}}^{(N)}$ of $\boldsymbol{\sigma}$ ($\equiv \boldsymbol{\sigma}^{(N)}$) that exactly recovers the communities,

$$\lim_{N \to \infty} \mathbb{P}\left(\hat{\boldsymbol{\sigma}}^{(N)} \in \{\pm \boldsymbol{\sigma}^{(N)}\}\right) = 1.$$

A first approach

In the assortative case:

- Solve the min-bisection problem:

$$\max \sum_{i,j} W_{ij} x_i x_j \qquad \text{subject to } \mathbf{x} \in \{\pm 1\}^N, \mathbf{1}^T \mathbf{x} = 0. \tag{1}$$

[1] Kim, C., Bandeira, A.S. and Goemans, M.X., 2018. Stochastic block model for hypergraphs: Statistical limits and a semidefinite programming approach. arXiv preprint arXiv:1807.02884.

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- SDP relaxation[1]:

$$\begin{aligned} \text{maximize} \quad & \sum_{1 \le i < j \le N} W_{ij} X_{ij} \\ \text{subject to} \quad & \sum_{1 \le i < j \le N} X_{ij} = 0, \\ & X_{ii} = 1 \text{ for all } i \in [N] \\ & \mathbf{X} \succeq 0. \end{aligned} \tag{2}$$

- Any solution $\mathbf{x}$ of (1) is a solution of (2) by taking $\mathbf{X} = \mathbf{x}\mathbf{x}^T$.

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Algorithm and main result

Step 1

Given $s \in \{\pm 1\}$ and $\mathbf{W}$, solve:

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Step 2

The optimal solution $\mathbf{X}^* = \sum_{i=1}^{N} \lambda_i \boldsymbol{v_i} \boldsymbol{v_i}^{\mathsf{T}}$ with $\lambda_1 \ge \cdots \ge \lambda_N$.

Step 3

Output $\hat{\boldsymbol{\sigma}} = \operatorname{sgn}(\boldsymbol{v_1})$

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Theorem

Suppose $(\boldsymbol{\sigma}, \mathbf{A}) \sim \mathrm{HSBM}(N, M, d, (\alpha_{\boldsymbol{\tau}}^{(m)}))$*, and let* $\mathbf{W}$ *be the aggregate similarity matrix of* $\mathbf{A}$*. When* $I > 1$*, the above algorithm with* $\mathbf{W}$ *and* $s = \operatorname{sgn}(\xi)$ *as inputs, exactly recovers* σ*.*

$$I = \sup_{\lambda \in \mathbb{R}} \sum_{m=1}^{M} \sum_{r=0}^{d-1} 2^{-(d-1)} \binom{d-1}{r} \alpha_{(r,d-r)}^{(m)} \left(1 - e^{-\lambda(d-1-2r)}\right)$$

Dual formulation

Primal problem:

$$\begin{aligned} \max \quad & \sum_{1 \le i < j \le N} W_{ij} X_{ij} \\ \text{subject to} \quad & X_{ii} = 1, \ \forall i \in [N] \\ & \langle \mathbf{X}, \mathbf{1}\mathbf{1}^T \rangle = 0, \\ & \mathbf{X} \succeq 0. \end{aligned} \quad \equiv \quad \begin{aligned} \min \quad & \langle \mathbf{W}', \mathbf{X} \rangle \\ \text{subject to} \quad & \langle \mathbf{A}_i, \mathbf{X} \rangle = 1, \ \forall i \in [N] \\ & \langle \mathbf{X}, \mathbf{J} \rangle = 0, \\ & -\mathbf{X} \preceq 0. \end{aligned}$$

where

$$(\mathbf{A}_i)_{jk} = 0 \text{ for } j \neq k, \text{ and } (\mathbf{A}_i)_{jj} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Lagrangian:

$$\mathcal{L}(\mathbf{X}, \mathbf{S}, \nu, \boldsymbol{d}) = \langle \mathbf{W}', \mathbf{X} \rangle - \langle \mathbf{S}, \mathbf{X} \rangle + \nu \langle \mathbf{X}, \mathbf{J} \rangle + \sum_{i=1}^{N} d_i \left(\langle \mathbf{A}_i, \mathbf{X} \rangle - 1 \right).$$

where $\mathbf{S} \succeq 0$.

Dual formulation

Dual objective:

$$\begin{aligned} g(\mathbf{S}, \nu, \boldsymbol{d}) &= \inf_{\mathbf{X}} \; \mathcal{L}(\mathbf{X}, \mathbf{S}, \nu, \boldsymbol{d}) \\ &= \inf_{\mathbf{X}} \; \langle \mathbf{W}' - \mathbf{S} + \nu \mathbf{J} + \operatorname{diag}(\boldsymbol{d}), \mathbf{X} \rangle - \sum_{i=1}^{N} d_i \\ &= \inf_{\mathbf{X}} \; \langle \mathbf{W}' - \mathbf{S} + \nu \mathbf{J} + \mathbf{D}, \mathbf{X} \rangle - \operatorname{trace}(\mathbf{D}) \\ &= \begin{cases} -\operatorname{trace}(\mathbf{D}) & \text{if } \mathbf{W}' - \mathbf{S} + \nu \mathbf{J} + \mathbf{D} = 0 \\ -\infty & \text{o.w.} \end{cases} \end{aligned}$$

Dual problem:

$$\begin{array}{rl} \max & -\operatorname{trace}(\mathbf{D}) \\ \text{subject to} & \mathbf{W}' - \mathbf{S} + \nu \mathbf{J} + \mathbf{D} = 0, \\ & \mathbf{S} \succeq 0 \end{array} \quad \equiv \quad \begin{array}{rl} \min & \operatorname{trace}(\mathbf{D}) \\ \text{subject to} & \mathbf{D} + \nu \mathbf{J} - \mathbf{W} \succeq 0. \end{array}$$

Dual certificate

$\mathbf{W}$: Observed aggregate similarity matrix

Lemma

Suppose there is a $N \times N$ diagonal matrix $\mathbf{D}$ such that $\mathbf{S} := \mathbf{D} + \mathbf{1}\mathbf{1}^T - \mathbf{W}$ satisfies:

$$\mathbf{S} \succeq 0, \quad \lambda_{N-1}(\mathbf{S}) > 0, \quad \text{and} \quad \mathbf{S}\boldsymbol{\sigma} = 0,$$

then $\mathbf{X}^* = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$ is the unique optimal solution to the SDP.

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Weak duality: **Optimality**

Let X be primal feasible and $X^* = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. Then

$$\begin{aligned}
\langle \mathbf{W}, \mathbf{X} \rangle &\leq \text{ trace } (\mathbf{D}) \\
&= \langle \mathbf{D}, \mathbf{X} \rangle = \langle \mathbf{D}, \mathbf{X}^* \rangle \qquad (\text{since } X_{ii} = X_{ii}^* = 1) \\
&= \langle \mathbf{W} + \mathbf{S} - \nu \mathbf{J}, \mathbf{X}^* \rangle = \langle \mathbf{W}, \mathbf{X}^* \rangle \quad (\text{since } \langle \mathbf{S}, \mathbf{X}^* \rangle = \boldsymbol{\sigma}^T (S\boldsymbol{\sigma}) = 0)
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Strong duality: **Uniqueness**

Let $\tilde{X}$ be an optimal solution and $X^* = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. Then

$$\begin{aligned}
\langle \mathbf{S}, \tilde{\mathbf{X}} \rangle &= \langle \mathbf{D} + \mathbf{J} - \mathbf{W}, \tilde{\mathbf{X}} \rangle = \langle \mathbf{D} - \mathbf{W}, \tilde{\mathbf{X}} \rangle \\
&= \langle \mathbf{D} - \mathbf{W}, \mathbf{X}^* \rangle \qquad (\langle \mathbf{W}, \tilde{\mathbf{X}} \rangle = \langle \mathbf{W}, \mathbf{X}^* \rangle \text{ and } \tilde{X}_{ii} = X^*_{ii} = 1) \\
&= \langle \mathbf{S}, \mathbf{X}^* \rangle = 0 \qquad (\text{since } \langle \mathbf{S}, \mathbf{X}^* \rangle = \boldsymbol{\sigma}^T (S\boldsymbol{\sigma}) = 0)
\end{aligned}$$

Since $\mathbf{S} \succeq 0$ and $\lambda_{N-1} > 0$, the $\text{Null}(\mathbf{S})$ is spanned by $\boldsymbol{\sigma}$ only.
$\tilde{\mathbf{X}} \succeq 0$ now implies that it should be a multiple of $\boldsymbol{\sigma}\boldsymbol{\sigma}^T$ as well.

Proof: Dual certificate

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Taking

$$D_{ii} := \sum_j W_{ij}\sigma_i\sigma_j,$$

easy to verify $\mathbf{S}\boldsymbol{\sigma} = 0$. Suffices to show

$$\mathbb{P}\left(\inf_{\mathbf{x}\perp\boldsymbol{\sigma}:\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{S}\mathbf{x} > 0\right) = 1 - o(1).$$

Proof: Bounds

Lemma

Let $\mathbb{E}\mathbf{W}$ *is the expected aggregate similarity matrix. Then*

$$\mathbf{x}^T\mathbf{S}\mathbf{x} \;\geq\; \min_i D_{ii} - \|\mathbf{W} - \mathbb{E}\mathbf{W}\|_2.$$

Proof: Bounds

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Proposition

Let $l > 1$. Then there exists a constant $\epsilon > 0$ dependent on model parameters such that for all $i \in [N]$,

$$\mathbb{P}(D_{ii} > \epsilon \log N) \geq 1 - o(N^{-1}).$$

Proposition

There exists a constant C such that

$$\mathbb{P}\left(\|\mathbf{W} - \mathbb{E}\mathbf{W}\|_2 \leq CM\sqrt{\log N}\right) \geq 1 - O(N^{-11}).$$

Proof: Main ingredients

- Rank-2 decomposition:

$$\mathbb{E}\mathbf{W} = \left(\frac{w_{\mathrm{in}} + w_{\mathrm{out}}}{2}\right)\mathbf{1}\mathbf{1}^T + \left(\frac{w_{\mathrm{in}} - w_{\mathrm{out}}}{2}\right)\boldsymbol{\sigma}\boldsymbol{\sigma}^T - w_{\mathrm{in}}\mathbf{I}_N,$$

where $w_{\mathrm{in}} = \mathbb{E}[W_{ij}|\sigma_i = \sigma_j]$ and $w_{\mathrm{out}} = \mathbb{E}[W_{ij}|\sigma_i \neq \sigma_j]$.

- Assortativity:

$$w_{\mathrm{in}} - w_{\mathrm{out}} \approx \frac{\log N}{2^{d-2}N}\xi.$$

Summary

- Multilayer HSBM $(\boldsymbol{\sigma}, \mathbf{A}) \sim \mathrm{HSBM}(N, M, d, (\alpha_{\boldsymbol{\tau}}^{(m)}))$
- Inhomogeneous hyperedge probabilities
- Assortative and disassortative cases
- Exact recovery using the similarity matrix $\mathbf{W}$

  SDP algorithm recovers the clusters exactly when $I > 1$.

Summary

- Multilayer HSBM $(\boldsymbol{\sigma}, \mathbf{A}) \sim \mathrm{HSBM}(N, M, d, (\alpha_{\tau}^{(m)}))$
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SDP algorithm recovers the clusters exactly when $I > 1$.

Future work

- Asymmetric: $\alpha_{({\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet})} \neq \alpha_{({\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet})}$
- Necessary conditions for exact recovery from $\mathbf{W}$
- For different hyperedge sizes d
- M, d depending on N

Thank you !!