Probabilistic forwarding of coded packets on networks

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Motivating example

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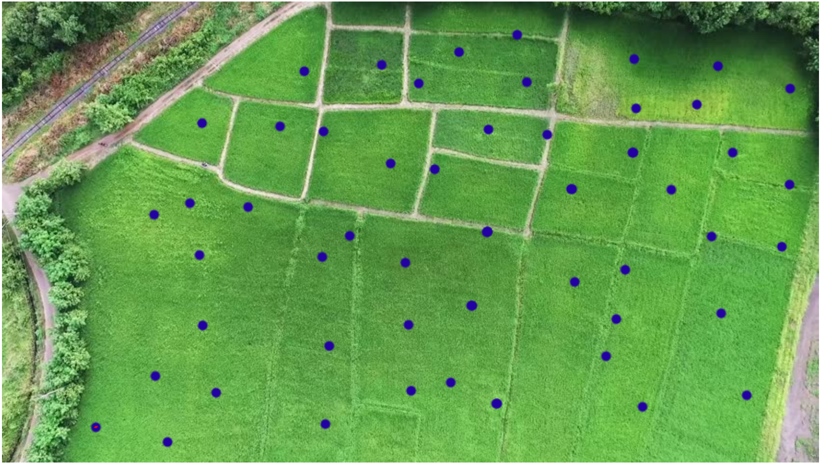


Sensors in a field

Motivating example

Sensors in a field

Motivating example

Sensors in a field

Update sensing parameters among all nodes

Network of IoT nodes

Over-the-air programming of the nodes

Motivating example

Sensors in a field

Update sensing parameters among all nodes

Network of IoT nodes

Over-the-air programming of the nodes

Broadcast information with the following constraints

Nodes:

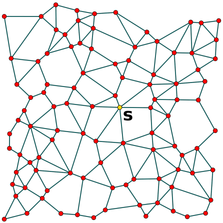
- Energy constrained
- Limited computational ability
- Limited knowledge of the network

Algorithm:

- Completely distributed
- Run in finite time

Abstraction

- **Broadcast over an ad-hoc network**



A network of nodes $\mathcal{G} = (V, E)$.
$|V| = N$.
A source node s has k data packets to be broadcast to all the other nodes in an ad-hoc network.

Flooding

- Each node forwards every received packet to all its one-hop neighbours.
- Subsequent receptions of the same packet are neglected.

- Total number of transmissions $= kN$.
- Wasteful [1]: nodes will receive same packet multiple times

[1] Y.-C. Tseng, S.-Y. Ni, Y.-S. Chen, and J.-P. Sheu, "The broadcast storm problem in a mobile ad hoc network," Wireless Networks, vol. 8, no. 2/3, pp. 153–167, 2002.

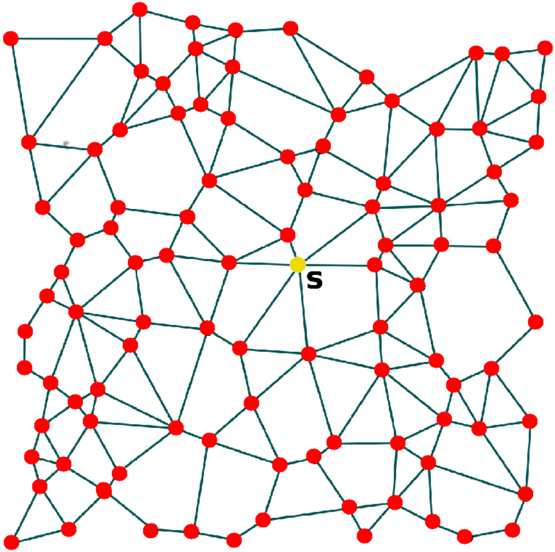
Probabilistic forwarding

- Retransmission probability p.
- Source transmits all k packets to all its neighbors.
- Each node, upon receiving packet $\#j$ for the first time, forwards it to all its neighbours with probability p; does nothing with probability $1 - p$.
- Subsequent receptions of the same packet are neglected.

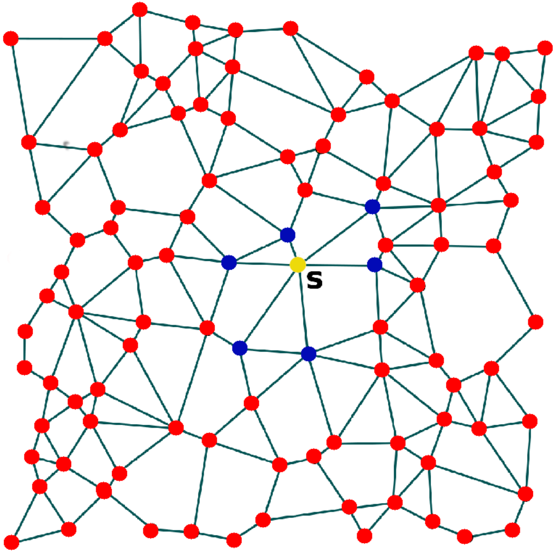
[1] Y. Sasson, D. Cavin, and A. Schiper, "Probabilistic broadcast for flooding in wireless mobile ad hoc networks," in Proc. IEEE Wireless Communications and Networking Conf. (WCNC) 2003, vol. 2, March 16–20, 2003, pp. 1124–1130.

[1] Z. J. Haas, J. Y. Halpern, and L. Li, "Gossip-based ad hoc routing," IEEE/ACM Trans. Networking, vol. 14, no. 3, pp. 479–491, 2006.

Probablistic forwarding



Probablistic forwarding



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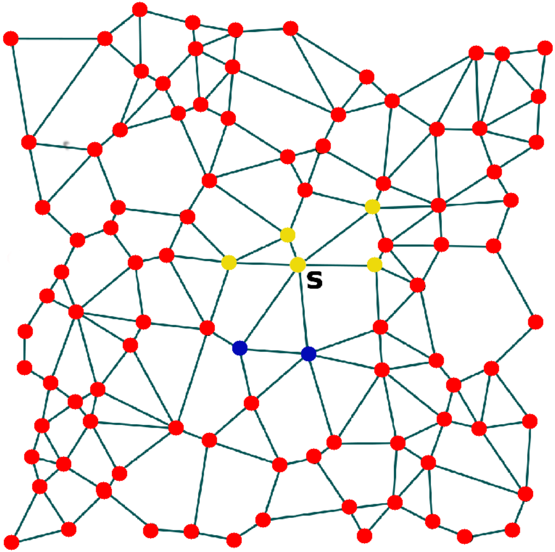

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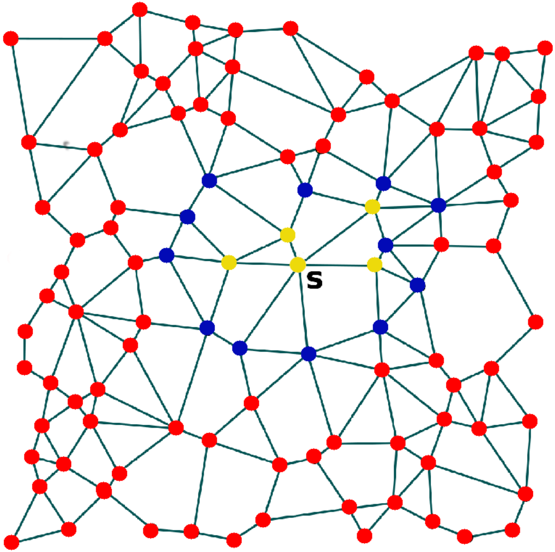

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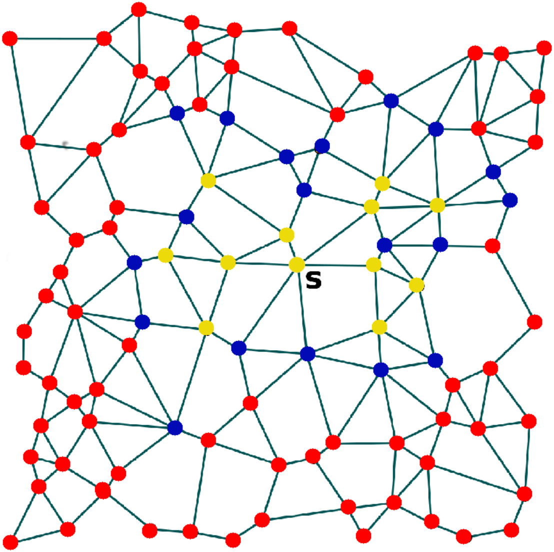

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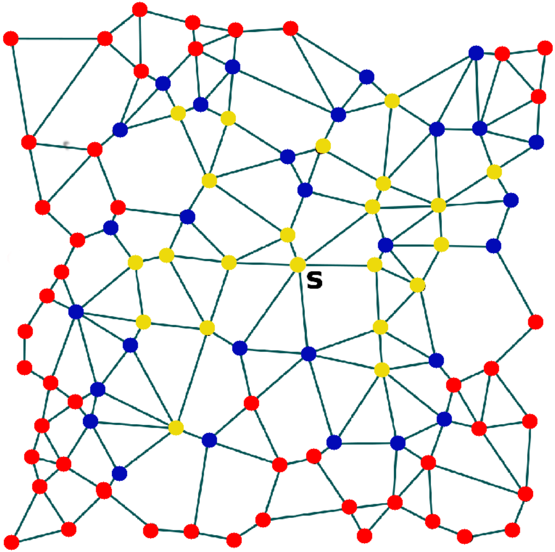

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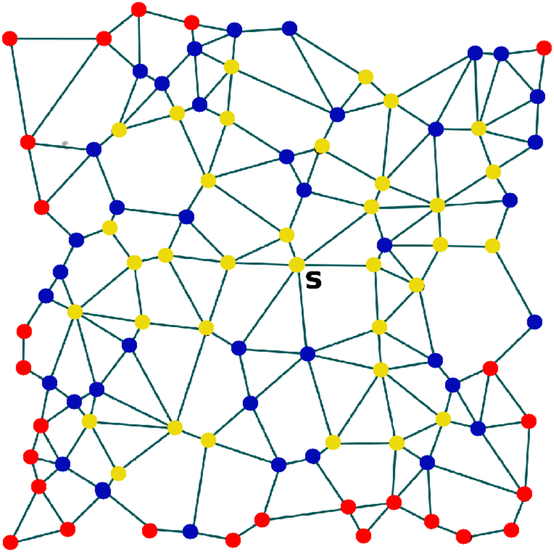

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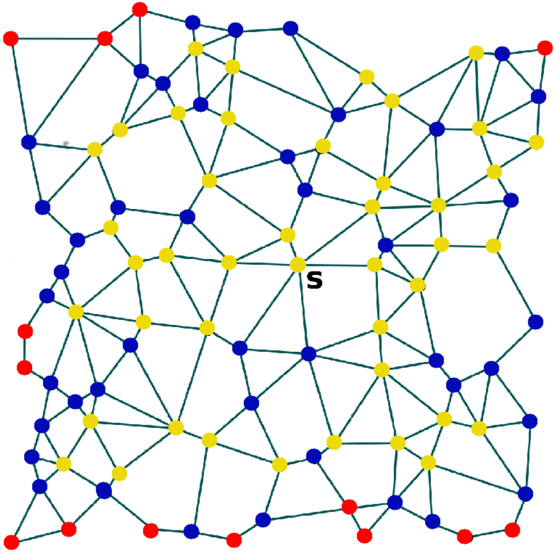

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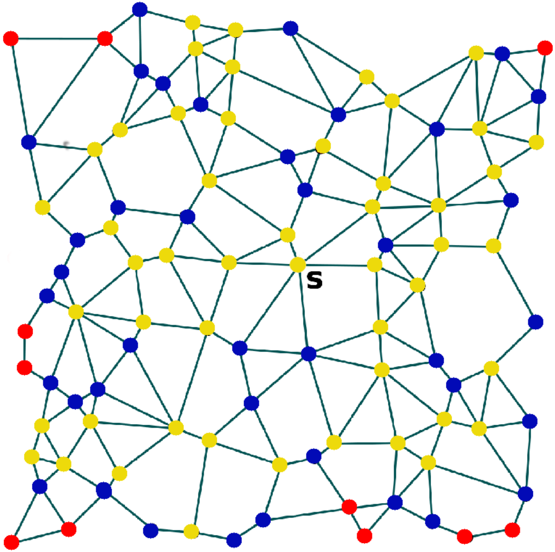

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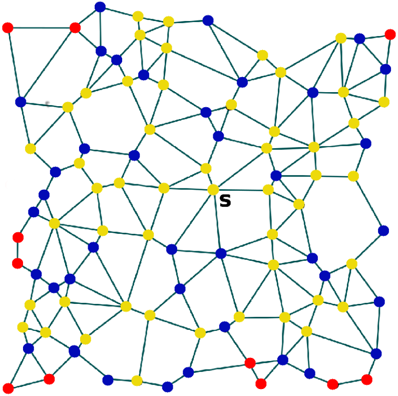


Lesser number of transmissions (kNp) compared to flooding (kN).

BUT

Information lost even if a single packet out of the k packets is not received.

Introducing Coded Packets

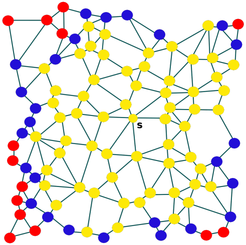
Coding:

- The source node encodes the k data packets into n coded packets using a Maximum Distance Separable (MDS) code.
- MDS code ensures that reception of any k of the n coded packets by any node, suffices to recover the original k data packets.

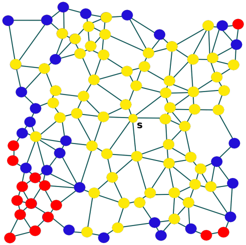
Probabilistic forwarding of coded packets:

- Source node transmits all n coded packets to its one-hop neighbours.
- Thereafter, the probabilistic forwarding protocol takes over. Each packet is forwarded independently of other packets and other nodes.
- Nodes which receive at least k out of n packets are termed successful receivers
- A near-broadcast is when the expected fraction of successful receivers is $\geq 1 - \delta$.

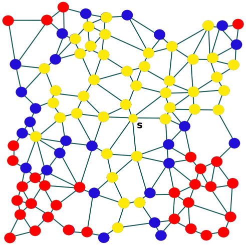
Illustration



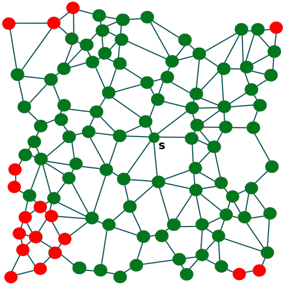
Packet 1



Packet 2



Packet 3



Successful receivers

- $k = 2, n = 3$
- (yellow) - Received and transmitted the packet.
- (blue) - Received but did not transmit the packet.
- (red) - Did not receive the packet.
- (green) - Successful receivers

Formal Problem Statement

Given:

- a connected graph $\mathcal{G}$ with N nodes
- number of data packets, k
- number of coded packets, n
- δ close to 0.
- retransmission probability p

Define

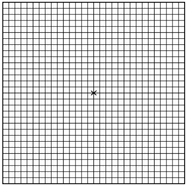
- $R_{k,n}$ = # nodes that receive at least k out of n coded packets

Want to find

- $p_{k,n,\delta}$ = minimum p such that $\mathbb{E}_p[\frac{1}{N} R_{k,n}] \geq 1 - \delta$
- $\tau_{k,n,\delta} = \mathbb{E}_{p_{k,n,\delta}}[\text{total \# transmissions over all } N \text{ nodes of } \mathcal{G}]$

On what graphs is coding along with probabilistic forwarding beneficial?

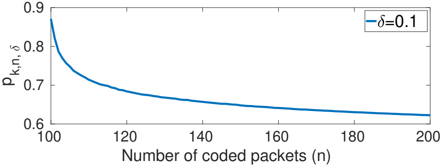
On well-connected graphs



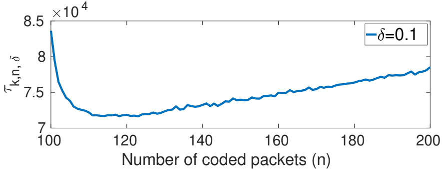
31 × 31 grid

1. $p_{k,n,\delta}$ decreases (to 0) as n increases; for fixed k and δ.
2. $\tau_{k,n,\delta} = \sum_{i=1}^{n} T_i$, where T_i is the expected number of transmissions of packet i.
    - Each term T_i decreases as n increases since $p_{k,n,\delta}$ decreases.
    - The number of terms in the above equation increases with n.

$k = 100$ packets and $\delta = 0.1$



Minimum forwarding probability

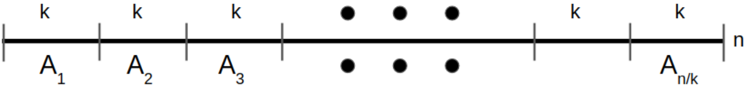


Expected total number of transmissions

Why $p_{k,n,\delta} \searrow 0$?

$R_{k,n}$ = # nodes that receive at least k out of n coded packets.

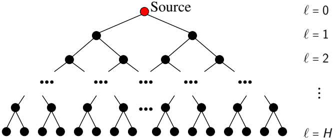
$p_{k,n,\delta} = \inf\{p \mid \mathbb{E}_p[\frac{1}{N} R_{k,n}] \geq 1 - \delta\}$

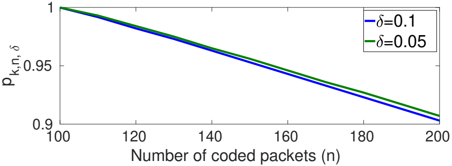


- For $j = 1, 2, \cdots, \lfloor \frac{n}{k} \rfloor$, let A_j be the event that the jth set of k coded packets is received by at least $1 - \delta/2$ fraction of the nodes.
- The events A_j are mutually independent and have the same probability of occurrence.
- For any $p > 0$, we have $P(A_j)$ being small but strictly positive.
- Hence, $P(\text{at least one } A_j \text{ occurs}) \geq 1 - \delta/2$ for all sufficiently large n, so that $\frac{1}{N} R_{k,n} \geq 1 - \delta/2$ with probability at least $1 - \delta/2$.
- Thus, for any $p > 0$, we have $p_{k,n,\delta} \leq p$ for sufficiently large n.

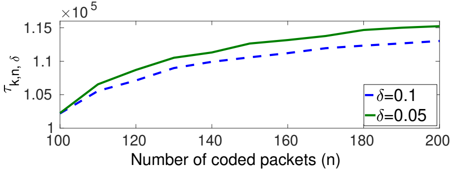
On trees

Rooted binary tree of height $H = 10$ with $k = 100$ and $\delta = 0.1$





Minimum forwarding probability



Expected total number of transmissions

- Large fraction of nodes on the leaves
- Unique path from the source to any node on the tree
- It can be shown that

$$p_{k,n,\delta} \approx c\left(\frac{k}{n}\right)^{\frac{1}{H-1}}$$

for some constant c dependent on H and δ.

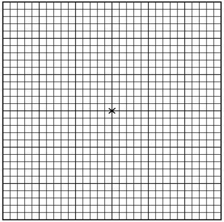
- This gives

$$\tau_{k,n,\delta} \approx n\,\frac{(2c)^H\left(\frac{k}{n}\right)^{\frac{H}{H-1}}-1}{2c\left(\frac{k}{n}\right)^{\frac{1}{H-1}}-1}$$

which is increasing in n.

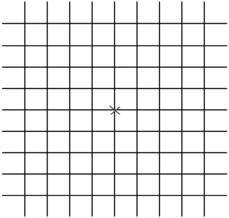
On the grid

Probabilistic forwarding on the $m \times m$ grid, Γ_m



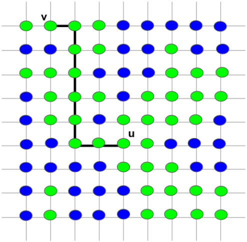
$\xrightarrow{m \to \infty}$

Probabilistic forwarding on the $\mathbb{Z}^2$ lattice



$\downarrow$

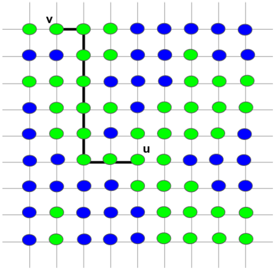
Site percolation on the $\mathbb{Z}^2$ lattice



We will see that the site percolation process on $\mathbb{Z}^2$ is a faithful model for the probabilistic forwarding mechanism on $\mathbb{Z}^2$.

Site percolation on $\mathbb{Z}^2$ - Definitions

- Associate each vertex (site) u of $\mathbb{Z}^2$ with a Bernoulli(p) random variable X_u. The vertex is open if $X_u = 1$; else closed.
- Two open sites u and v are said to be connected by an open path ($u \longleftrightarrow v$), if there is a path of open sites from u to v.



Site percolation on $\mathbb{Z}^2$- Transmitters

- Associate each vertex (site) u of $\mathbb{Z}^2$ with a Bernoulli(p) random variable X_u. The vertex is open if $X_u = 1$; else closed.
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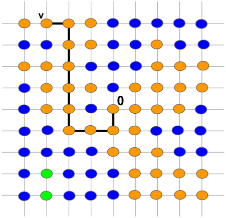
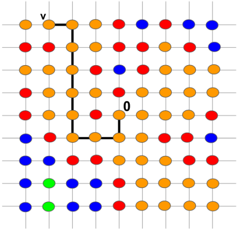


- Probabilistic forwarding of a single packet over $\mathbb{Z}^2$ is modelled by site percolation on $\mathbb{Z}^2$ conditioned on the origin $\mathbf{0}$ being open.
- Nodes transmitting the j**th packet** (for fixed $j \in [n]$) may be viewed as open sites of a site percolation process which are connected to the origin by an open path. Call this cluster of nodes as C_j^o.
- The total number of transmissions is simply $\sum_{j=1}^{n} |C_j^o|$.

Site percolation on $\mathbb{Z}^2$ – *Receivers*

- The boundary, ∂C_j^o, is the set of all closed sites which are adjacent to a site in C_j^o.
- The set $C_j^{o+} := C_j^o \cup \partial C_j^o$ is called the extended cluster of the origin.



For site percolation on $\mathbb{Z}^2$, there exists $p_c \in (0,1)$ s.t. for $p > p_c$,

- There exists a unique infinite open cluster (IOC), C, almost surely. $p_c \approx 0.59$ for site percolation.
- Hence, there also exists a unique infinite extended cluster (IEC), $C^+ = C \cup \partial C$, a.s..
- $\theta(p) :=$ percolation probability, i.e., $\mathbb{P}(\mathbf{0} \in C)$
  $\theta^+(p) := \mathbb{P}(\mathbf{0} \in C^+)$

Lemma: $\theta^+(p) = \frac{\theta(p)}{p}$.
Proof: $\{\mathbf{0} \in C\} = \{\mathbf{0} \in C^+ \text{ and } \mathbf{0} \text{ is open}\}$

Ergodic theorems

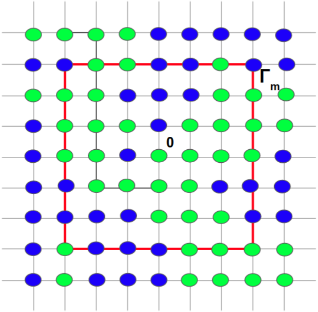
For $p > p_c$, if C is the unique IOC and C^+ is the unique IEC, then,

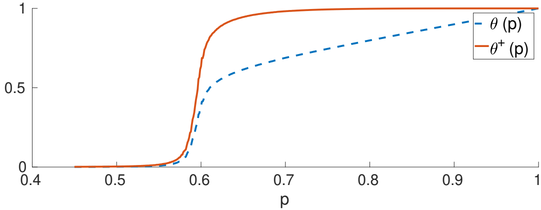
$$\lim_{m\to\infty} \frac{|C \cap \Gamma_m|}{m^2} = \theta(p) \qquad a.s.$$

Similarly,

$$\lim_{m\to\infty} \frac{|C^+ \cap \Gamma_m|}{m^2} = \theta^+(p) \qquad a.s.$$

Using DCT, expected values also converge.





Site percolation and probabilistic forwarding

- Prob. forwarding of a single packet over $\mathbb{Z}^2$ is modelled by site percolation on $\mathbb{Z}^2$ conditioned on the origin $\mathbf{0}$ being open.
- n pkts $\leftrightarrow$ n independent site percolations with $\mathbf{0}$ open in all.
- $\mathcal{R}_{k,n}^{pr.fwd}(\Gamma_m) :=$ $\{$sites in Γ_m that receive at least k out of n pkts$\}$
- We are interested in finding

$$p_{k,n,\delta} = \min\left\{p \;\middle|\; \mathbb{E}_p\left[\frac{1}{m^2}\left|\mathcal{R}_{k,n}^{pr.fwd}(\Gamma_m)\right|\right] \geq 1-\delta\right\}$$

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Theorem

For $p > p_c$, we have

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}_{k,n}^{pr.fwd}(\Gamma_m)\right|\right] = \sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^+(p))^{t+j}(1-\theta^+(p))^{n-j}.$$

Total transmissions

Consider the transmissions of a single packet on $\mathbb{Z}^2$

- C^o := open cluster containing $\mathbf{0}$ (set of transmitters)
- The expected number of transmissions on a large grid, Γ_m, is given by $\mathbb{E}\big[|C^o \cap \Gamma_m| \;\big|\; \mathbf{0} \text{ is open}\big]$.

Proposition

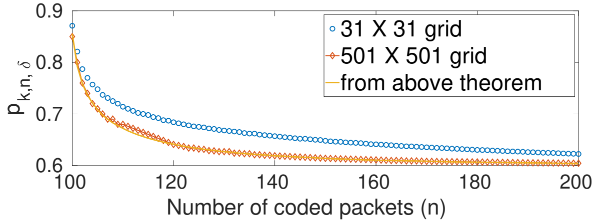
For $p > p_c$, we have

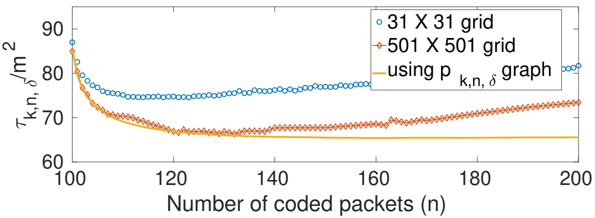
$$\lim_{m\to\infty} \frac{1}{m^2}\mathbb{E}\big[|C^o \cap \Gamma_m| \;\big|\; \mathbf{0} \textit{ is open}\big] = \frac{\theta(p)^2}{p}.$$

- Hence, for n coded packets, with each packet being transmitted with probability $p_{k,n,\delta} > p_c$, we obtain

$$\tau_{k,n,\delta} \approx nm^2 \frac{\theta^2(p_{k,n,\delta})}{p_{k,n,\delta}}.$$

Comparison with simulation results





Proof sketch

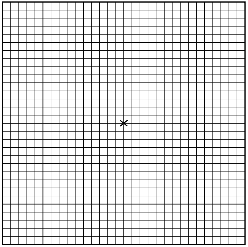
We want to show,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}^{pr.fwd}_{k,n}(\Gamma_m)\right|\right] = \sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^+(p))^{t+j}(1-\theta^+(p))^{n-j}.$$

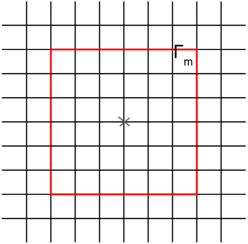
Step 1: $\mathcal{R}^{pr.fwd}_{k,n}(\mathbb{Z}^2) := \{\text{sites that receive at least } k \text{ out of } n \text{ pkts on } \mathbb{Z}^2\}$

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}^{pr.fwd}_{k,n}(\Gamma_m)\right|\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}^{pr.fwd}_{k,n}(\mathbb{Z}^2)\cap\Gamma_m\right|\right]$$

Probabilistic forwarding on the $m \times m$ grid, Γ_m



Probabilistic forwarding on the $\mathbb{Z}^2$ lattice



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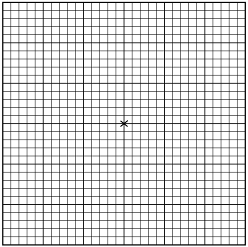

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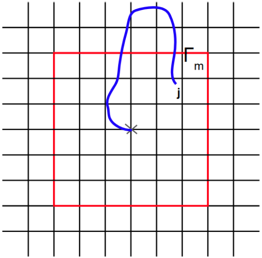

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Step 1: $\mathcal{R}_{k,n}^{pr.fwd}(\mathbb{Z}^2) := \{$sites that receive at least k out of n pkts on $\mathbb{Z}^2\}$

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}_{k,n}^{pr.fwd}(\Gamma_m)\right|\right] = \lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}_{k,n}^{pr.fwd}(\mathbb{Z}^2)\cap\Gamma_m\right|\right]$$

Step 2: Let $\mathcal{R}_{k,n}^{perc}(\mathbb{Z}^2) := \{$sites that are in at least k out of n C^+s$\}$
Using ergodicity, we show that for $p > p_c$,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{\left|\mathcal{R}_{k,n}^{perc}(\mathbb{Z}^2)\cap\Gamma_m\right|}{m^2}\right] = \underbrace{\sum_{j=k}^{n}\binom{n}{j}(\theta^+(p))^{j}(1-\theta^+(p))^{n-j}}_{\equiv\ \theta_{k,n}^+(p)}.$$

Proof sketch

We want to show,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}^{pr.fwd}_{k,n}(\Gamma_m)\right|\right] = \sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^+(p))^{t+j}(1-\theta^+(p))^{n-j}.$$

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Recall:

- $\theta^+(p) = \mathbb{P}(\mathbf{0}\in C^+)$
- $\lim_{m\to\infty} \frac{|C^+\cap\Gamma_m|}{m^2} = \theta^+(p) \qquad a.s.$

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We want to show,

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Step 3: Carefully relate $\mathcal{R}^{pr.fwd}_{k,n}(\mathbb{Z}^2)$ and $\mathcal{R}^{perc}_{k,n}(\mathbb{Z}^2)$ to obtain,

$$\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m^2}\left|\mathcal{R}^{pr.fwd}_{k,n}(\mathbb{Z}^2)\cap\Gamma_m\right|\right] = \sum_{t=k}^{n}\sum_{\substack{T\subseteq[n]:\\|T|=t}}\theta^+_{k,t}(p)(\theta^+(p))^{t}(1-\theta^+(p))^{n-t}.$$

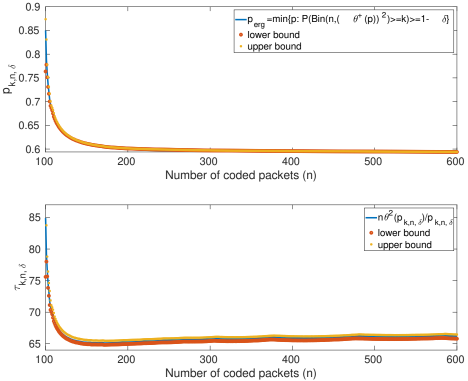
Some bounds

It can be shown that

$$\sum_{t=k}^{n}\sum_{j=k}^{t}\binom{n}{t}\binom{t}{j}(\theta^+(p))^{t+j}(1-\theta^+(p))^{n-j} = Pr(Y \geq k)$$

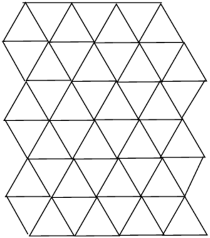
where $Y \sim Bin\left(n, (\theta^+(p))^2\right)$. So, for large grids, we have

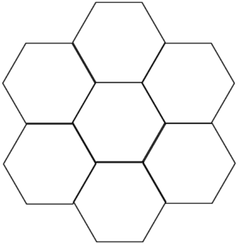
$$p_{k,n,\delta} = \inf\{p : Pr(Y \geq k) \geq 1-\delta\} \text{ and } \tau_{k,n,\delta} \approx nm^2\frac{\theta^2(p_{k,n,\delta})}{p_{k,n,\delta}}.$$



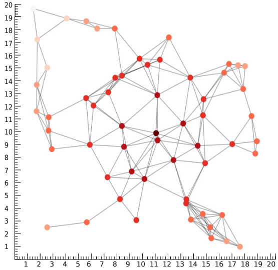
Other graphs

Lattices- triangular, hexagonal.

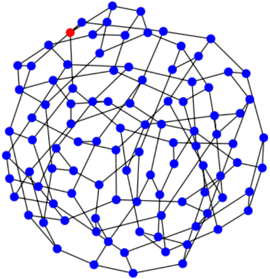




Random geometric graphs



Random regular graphs



Thank you

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